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Per 1875 d 42

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THE

# OXFORD, CAMBRIDGE, AND DUBLIN
# MESSENGER OF MATHEMATICS,

A JOURNAL SUPPORTED BY

JUNIOR MATHEMATICAL STUDENTS,

AND

CONDUCTED BY A BOARD OF EDITORS

COMPOSED OF

MEMBERS OF THE THREE UNIVERSITIES.

---

"Ἄγγελον ἐσλὸν ἔφα τιμὰν μεγίσταν πράγματι παντὶ φέρειν·
Αὔξεται καὶ Μάθησις δι' ἀγγελίας ὀρθάς.—PINDAR.

---

VOL. I.

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1862.

ON the conclusion of a volume we revert to the anticipations which attended the publication of its first number, with a view to consider how far the objects, of which our *Introduction* was the exponent, have been up to this point attained. Thanks then are due to our contributors, whether they have chosen the path of original investigation, or the compilation of articles to be read in connexion with standard elementary works; and special thanks to those Senior Students who have rendered the most advanced results accessible by their lucid and compendious introductions to recent offshoots of Mathematical science. But with regard to the increasing divergence of Oxford, Cambridge, and Dublin Mathematicians, and the advantages of an undivided 'English school,' it may be remarked that the uniformity which results from the predominance of one branch of contributors to the comparative exclusion of the rest is not by any wise to be desired. We regret to state that of the number of articles placed at our disposal for this first volume no less than ninety-one per cent. came from one University.

A word still remains to be said: this Journal, having existed for scarcely a year, is not yet so well known as we hope it deserves to be. It rests then with our supporters to increase by their influence a somewhat deficient circulation and thus render a service which we hope will not require to be again asked for when the *Messenger* shall have completed a second stage.

Cambridge, *October*, 1862.

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## Errata.

Page 5, in the note, *for* Frost's *Newton*, *read* Evan's *Newton*.
7, line 6, *for* whenever, *read* wherever.
8, 11, *for* similar to $OP'$, *read* similar to $OGP$.
48, 23, *for* $2ln\cos A$, *read* $2mn\cos A$.
96, 20, *for* $2\rho$, *read* $\rho$.
96, 22, *for* $\dfrac{\frac{1}{4}}{\text{latus rectum}}$, *read* $\dfrac{4}{\text{latus rectum}}$.
102, in the first determinant, *for* $\sec 2C$, *read* $\sin 2C$.
104, line 23, *for* affected, *read* effected.
141, 2, *for* $\sin\phi a\cos\widehat{\delta}$, *read* $\sin\phi\cos\widehat{a\delta}$.
142, 16, *for* $P=\rho$, *read* $OP=\rho$.

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# The Oxford, Cambridge, and Dublin
# MESSENGER OF MATHEMATICS.

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## INTRODUCTION.

In inviting the cooperation of members of the Universities in supporting a Journal of Mathematics for Junior Students, it will be well to state distinctly what objects the promoters of the Magazine propose to themselves.

It is impossible to discuss, in the limited space of an Introduction, the relative importance of original investigation, and the study of systematic treatises. But it may be remarked, that the solving of problems, unless they are of a high order of originality and conception, can scarcely be looked upon as original investigation at all. Analytical methods are so complete and so uniform in their modes of application to the problems which usually come before Undergraduates, that the first, and incomparably the most important in an educational view, of the three operations of the mind which are performed in solving a problem, has degenerated into a *memoria technica* of formulæ. The highest mathematical operation is the reduction of the elements of a problem to mathematical language, and the expression of its conditions in equations: and the possibility of this expression will remain the most wonderful proof of the importance and generality of mathematical science, and perhaps the most marvellous monument of the powers of the human mind. What value can this operation possess when it is performed mechanically—nay when the advice comes to be given with the authority of a tutor—"Shut your eyes and write down your equations"? The operation of solving these equations is of entirely second-rate value; and yet it occupies—and the limited range of low subjects makes it necessary that it should occupy—by far the most important place in a mathematical education. The interpretation of the solution rarely comes before students in other than the simplest forms.

Hence it would appear, that if original work is of any importance to junior students, the field for it must be sought rather in problems of their own creation, than in such as follow the accepted type so closely as to deserve rather the name of examples than of problems. We hope, therefore, that we shall be able to induce such students to attempt original investigation in their favourite branches of mathematics, and that the distinctness of conception, and the exercise of imagination required for such work will be found to react on themselves with profit in University examinations.

But we by no means wish to close our pages against what is written with very different objects. It will be of the highest importance to introduce to the notice of students the gradual development of various branches of mathematics, that they may learn to contemplate mathematical science in its scientific and historical unity. Most men crave for this knowledge; and this craving is in itself a proof that there is some deficiency in our curriculum. The most refined conceptions of analysis are laid as almost axiomatic before the young student, and he receives them not with the vigorous faith of one who has seen previous generalizations merge into others of still higher generality, but with the passive faith of one incapable of using a single argument for or against the assumptions. The unity, neatness, and generality of our analysis is often gained at the expense of soundness of conviction with respect to the ground-work of the notation. We may use as an illustration the almost universal dislike which our students feel towards questions on the relations between the units employed in analysis. Now this Magazine may be of great use in remedying this deficiency, if mathematicians of higher standing will furnish us with papers on these subjects for the benefit of their younger brethren.

So admirable a model of an analytical branch of mathematics historically developed, has been given by Mr. Todhunter in his *History of the Calculus of Variations*, that the demand for similar works inevitably follows; and though works of that extent will be more properly published elsewhere, historical papers on more confined subjects, or with less of detail, will form a most valuable addition to our table of contents.

Again, when we invite mathematicians to send us elementary papers on comparatively new branches of mathematics, which are scarcely included in the ordinary course at the Universities, it is not without a profound conviction of the value of progressive studies to the student. The paper on Determinants which we publish in the present number, is an example of a subject which has been for the first time laid before Cambridge students in three places almost simultaneously, and we hope it may be followed by similar papers on the progress which analysis has made recently in elimination and transformation of equations.

In those branches also of mathematics, such as Astronomy, in which young students can never hope to accomplish anything, and which seem to them to leave nothing to be accomplished, it would be most interesting to a considerable number of readers, if some competent person would occasionally review the state of the science and the problems on which our professors at home and abroad are engaged. What for example is the state of the problem respecting the residual irregularity in the moon's motion? What is the exact discovery in the integration of the equation for the velocity of sound? What are the present problems in light, if they can at all be explained to us? What is this allpowerful new Calculus of Quaternions, the modern equivalent, as it would seem, of the Pythagorean

*τετρακτύν,*
*Παγὰν ἀενάου φύσεως ῥιζώματ' ἔχουσαν.*

We would most earnestly solicit information on such subjects as these; for which no magazine that previously existed, is at all adapted.

Physico-mathematical theories, such as proceed to argue on a hypothetical molecular constitution of bodies, with their development from the subtle conceptions of the ancient Greek school to those of Boscovich and Challis, would interest and be of great value to a large class of readers. Criticisms also on mathematical methods and books in common use; on the present state of mathematical education in the Universities, and its prospects; and similar papers may be of considerable value.

Further, since it may be said with some show of justice, that in opening our pages to detached and isolated problems which are remarkable for neatness and ingenuity, new methods or new theory, we are pandering to the riddle-solving and puzzle-working tastes of our contemporaries, we must admit, that to some extent, this charge is true; but good riddle-solving is not without its uses, and clever puzzle-working has special fascinations for mathematicians of a certain stamp. We shall endeavour, however, to exercise a somewhat strict censorship over articles of this kind that are sent to us; knowing well the most difficult work of the Editors will be in the selection from contributions of this class.

Lastly, the mathematicians of Oxford, Cambridge, and Dublin are more and more widely separating in style and selection of subjects; and this bids fair to be a serious evil. Let us have an English school of mathematics by all means, but sub-divisions in that school are simply an evil. We conceive that it will be no small advantage to bring together, by ever so slight a bond, the young members of the different Universities who may one day meet on the most exalted platforms of science, and thus early introduce them to one another.

But our Introduction has exceeded its assigned bounds, and we must conclude. We furnish an audience, eager, attentive and intelligent; we have no fear that our requests will fall to the ground unheard or unheeded.

# THE EQUIANGULAR SPIRAL, ITS CHIEF PROPERTIES PROVED GEOMETRICALLY.

Of the following propositions, those only which give geometrical constructions for the centres of gravity of a spiral arc and area present results which will probably be new to the reader. But the other propositions are added as it is thought convenient to have geometrical demonstrations of the principal properties of this beautiful curve collected into a continuous paper, and it is hoped it will be found at once profitable and interesting to view the curve under the somewhat novel aspect under which it is here presented.

Def. *Any plane curve proceeding from a fixed point (which is called the pole) and such that the arc intercepted between this point and any other whatever on the curve is always similar to itself, is called an equiangular spiral.**

We have adopted the above definition rather than that usually given, as the demonstrations we are about to present are founded on the property of continual similarity, rather than on that from which the curve takes its name. And we think that the property we have adopted is the more natural one by which to define the curve, for it is (so to speak) *more thoroughly intrinsic*, involving only the form of the curve without the accessories of tangents and radii vectores.

We proceed to deduce from our definition the property by which the curve is usually defined and from which it gains its name.

---

* Of the various definitions of similar figures the most convenient to adopt is that given in Frost's *Newton*, p. 11, viz.:

"Two curves are said to be similar, when there can be drawn in them two distances from two points similarly situated, such that, if any two other distances be drawn equally inclined to the former, the four are proportional."

Prop. I. *The tangent at any point of an equiangular spiral makes a constant angle with the polar radius vector.*

For, let $OP$, $OQ$ (fig. 1) be any two polar radii in the spiral $OKP$; $OP'$, $OQ'$ two other radii making equal angles $POP'$, $QOQ'$ with the former two.

Then since $OKP$, $OKQ$ are similar figures, and $OP'$, $OQ'$ make equal angles with the corresponding lines $OP$, $OQ$; therefore $OPP'$, $OQQ'$ are similar triangles, and therefore the angles $OPP'$, $OQQ'$ are equal; therefore, ultimately, when $P'$, $Q'$ move up to and coincide with $P$, $Q$, the tangents at $P$ and $Q$ will make equal angles with the polar radii vectores.

And these are any two points on the curve.

Therefore, &c. Q.E.D.

Def. The constant angle between the tangent and radius vector is called the *angle of the spiral.* We shall in future always denote it by $\alpha$.

Prop. II. *If upon any polar radius vector* OP, *a triangle* OPQ *be described similar to a given triangle, the locus of the vertex* Q *will be a spiral similar to the original spiral* (fig. 2).

For, let $OP$, $OP'$ be any two polar radii, $OQP$, $OQ'P'$ the triangles described upon them. Then since these are similar, we have

$$OP : OP' = OQ : OQ'.$$

Again, since the angles $POQ$, $P'OQ'$ are equal, add (or subtract) the angle $QOP'$, then the angles $POP'$, $QOQ'$ are equal; and we have shewn that

$$OP : OP' = OQ : OQ';$$

therefore $PP'$, $QQ'$ are similar curves, $OP$, $OQ$ being corresponding lines.

Therefore, &c. Q.E.D.

Prop. III. *If from the pole* O *there be drawn a line* OQ *at right angles to the radius vector* OP, *so as to meet the normal at* P *in* Q, *then the locus of* Q *is the evolute of the* *spiral, and is a similar spiral, and* Q *is the centre of curvature at the point* P (fig. 3).

For the angle $OPQ$ is the complement of the angle of the spiral; therefore the angle $OQP$ is equal to the angle of the spiral; therefore the triangle $OQP$ is always similar whenever $P$ be taken.

Therefore (Prop. II.) the locus of $Q$ is a similar spiral, and since $OQP$ is equal to the angle of the spiral, therefore $QP$ is the tangent at $Q$ to the locus of $Q$; therefore the locus of $Q$ is the envelope of the normal $PQ$, or is the evolute of the curve, and $Q$ is the centre of curvature at $P$. Q.E.D.

Cor. I. *Produce the radius vector* PO *to* V, *making* OV *equal to* OP, *then* PV *is the chord of curvature through* O.

For since $Q$ is the centre of the circle of curvature, and $QO$ bisects $PV$ at right angles; therefore $P$ being on the circle, so also is $V$. Therefore $PV$ is the chord.

Cor. II. If $OY$ be the perpendicular on the tangent at $P$, we have

$$OY = OP \sin\alpha,$$

$$OP = PQ \sin\alpha,$$

or, with the usual notation in analytical methods,

$$p = r \sin\alpha, \quad r = \rho \sin\alpha.$$

Prop. IV. Problem. *To find the length of the arc from the pole to any fixed point.*

Let $OP$, $OP'$ (fig. 4) be two polar radii very near to one another, then since $OKP$, $OKP'$ are similar curves, (Def.)

$$\text{arc } OKP : \text{arc } OKP' = OP : OP';$$

therefore $$\text{arc } OKP : \text{arc } PP' = OP : OP' - OP.$$

Draw $PN$ perpendicular on $OP'$, then, ultimately, when $PP'$ is indefinitely diminished $OP' - OP = P'N$; therefore, ultimately,

$$\text{arc } OKP : PP' = OP : P'N,$$

alt. $$\text{arc } OKP : OP = PP' : P'N = \sec\alpha : 1;$$

therefore $\text{arc } OKP = OP \sec\alpha$,

or the length of the arc between the pole and any fixed point is equal to the product of the length of extreme radius by the secant of the angle of the spiral.

PROP. V. PROBLEM. *To find the centre of gravity of a homogeneous arc of an equiangular spiral between the pole and any fixed point.**

Let $OKP$ (fig. 5) be the spiral, $OP$, $OP'$ two polar radii near to one another. Suppose any point $G$ the centre of gravity of $OKP$. Join $OG$, $GP$, and on $OP'$ describe the triangle $OG'P'$ similar to $OP'$. Then since $G$, $G'$ are corresponding points in the similar figures $OKP$, $OKP'$; therefore $G'$ is the centre of gravity of $OKP'$. Therefore $GG'$ produced must pass through the centre of gravity of $PP'$. Provided we ultimately make $P'$ move up to and coincide with $P$, we may take $P$ as the centre of gravity of $PP'$, and ultimately, when $P$, $P'$ coincide, $G$, $G'$ will coincide, and $GG'$ will become the tangent at $G$ to the locus of $G$; therefore the tangent at $G$ to the locus of $G$ passes through $P$. But the locus of $G$ is by Prop. II. a similar spiral; therefore the angle $OGP$ is the supplement of the angle of the spiral.

Again, since $G$, $G'$ are the centres of gravity of the arcs $OKP$, $OKP'$ and $P$ the centre of gravity (ultimately) of their difference, we have ultimately

$$PG \text{ arc } PP' = GG' \text{ arc } OKP.$$

But by similar figures

$$PP' : GG' = OP : OG,$$

---

* We have given analytical not synthetical proofs of this proposition and the seventh, for a reason similar to that which probably influenced Euclid in framing his proof of the last proposition of Book IV. of the Elements. In most cases where the construction is somewhat complicated the synthesis will of necessity appear artificial and tentative, insomuch that it seems better to present the reader with the analysis itself and leave him to draw out for himself a synthetical demonstration.

and by Prop. IV. arc $OKP = OP \sec\alpha$;

therefore $PG = OG \sec\alpha$.

Draw $PN$ perpendicular on $OG$; then

$$GN = PG \cos\alpha = OG,$$

therefore $OG = \frac{1}{2} ON$.

In $OP$ take $OR = \frac{1}{2} OP$,

then the circle on $OR$ as diameter passes through $G$; therefore $G$ is the point of intersection of the circle on $OR$ as diameter with the circular segment on $OP$ as chord containing an angle supplementary to the angle of the spiral.

Thus $G$ is determined. Q.E.F.

COR. If $O$ be the pole, $OP$ the extreme radius vector of a series of equiangular spirals of different angles, the centres of gravity of their arcs all lie on the circle passing through $O$, and having its centre in $OP$ at a distance from $O$ equal to $\frac{OP}{4}$. (S. H. Examination, Jan. 18, 1859).

PROP. VI. PROBLEM. *To find the area swept out by the radius vector in describing the spiral between the pole and any given point.*

Let $OP$, $OP'$ (fig. 4) be two radii near to one another, then since $OKP$, $OKP'$ are similar figures, their areas are in the duplicate ratio of the homologous sides; therefore

$$\text{area } OKP : \text{area } OKP' = OP^2 : OP'^2,$$

and therefore area $OKP$ : area $POP' = OP^2 : OP'^2 - OP^2$;

therefore $$\text{area } OKP = OP^2 . \frac{\text{area } POP'}{OP'^2 - OP^2},$$

this is true always, and therefore true in the limit when $P'$ moves up to and ultimately coincides with $P$. Draw $PN$ perpendicular to $OP'$, then

$$\text{area } POP' = \tfrac{1}{2} OP'.PN \text{ ultimately},$$

also $OP'^2 - OP^2 = 2\, OP'.P'N$ ultimately;

therefore $\text{area } OKP = OP^2 . \text{limit of } \dfrac{\frac{1}{2} OP'.PN}{2 OP'.P'N} = \frac{1}{4} OP^2 \tan\alpha,$

or the area swept out by the radius vector between the pole and any given point is equal to one-fourth of the product of the square of the extreme radius by the tangent of the angle of the spiral.

**Prop. VII. Problem.** *To find the centre of gravity of the area swept out by the radius vector between the pole and any given point in the equiangular spiral.*

Let $OP$, $OP'$ (fig. 5) be two adjacent radii. Assume $G$ for the centre of gravity of the area $OKP$, then, as in Prop. V., $G'$ similarly situated in $OKP'$ will be the centre of gravity of the area $OKP'$.

Take $OH$ (fig. 6) in $OP$ equal to $\frac{2}{3} OP$, then ultimately $H$ is the centre of gravity of the evanescent area $OPP'$; therefore, as in Prop. V., the angle $OGH$ is supplementary to the angle of the spiral, and

$$GG' \text{ area } OKP = GH \text{ area } OPP'.$$

Now $\text{area } OKP = \frac{1}{4} OP^2 \tan\alpha$, Prop. VI.,

$\text{area } OPP' = \frac{1}{2} OP.PP' \sin\alpha,$

also $GG' : PP' = OG : OP$ by similar triangles;

therefore $OG \sec\alpha = 2\,GH$;

therefore, if we draw $HN$ perpendicular on $OG$, we shall have

$$GN = GH \cos\alpha = \tfrac{1}{2} OG;$$

therefore $OG = \frac{2}{3} ON.$

In $ON$ take $OR = \frac{2}{3} OH = \frac{4}{9} OP,$

then the circle on $OR$ as diameter passes through $G$.

Therefore $G$ is the point of intersection of the circle on $OR$ as diameter with the circular segment on $OH$ as chord containing an angle supplementary to the angle of the spiral, where

$$OH = \tfrac{2}{3} OP, \quad OR = \tfrac{4}{9} OP.$$

Thus $G$ is determined. Q.E.F.

**Prop. VIII. Problem.** *To find the moment of inertia of the arc of an equiangular spiral about a perpendicular through its pole to its plane.*

Suppose the mass of an unit of length to be unity.

Let $OP$, $OP'$ be any two radii, then since $OKP$, $OKP'$ are similar curves, we have

$$\frac{\text{moment of inertia of } OKP}{OP^3} = \frac{\text{moment of inertia of } OKP'}{OP'^3},$$

and therefore $$= \frac{\text{moment of inertia of } P'P}{OP'^3 - OP^3},$$

and this will be true in the limit when $P'$ coincides ultimately with $P$; therefore

$$\frac{\text{moment of inertia of } OKP}{OP^3} = \frac{OP^2 . P'P}{3\,OP^2 . P'N} = \tfrac{1}{3}\sec\alpha,$$

or $$\text{moment of inertia of } OKP = \tfrac{1}{3} OP^3 \sec\alpha;$$

therefore the moment of inertia of any arc from the pole about a perpendicular to its plane through the pole is one-third of the product of the cube of the extreme radius by the secant of the angle of the spiral.

**Prop. IX. Problem.** *To find the moment of inertia of the area of an equiangular spiral about a perpendicular to its plane through the pole.*

Suppose the mass of an unit of area to be unity.

Let $OP$, $OP'$ (fig. 4) be any two radii, then since $OKP$, $OKP'$ are similar curves, we shall have

$$\frac{\text{moment of inertia of } OKP}{OP^4} = \frac{\text{moment of inertia of } OKP'}{OP'^4},$$

and therefore $$= \frac{\text{moment of inertia of } POP'}{OP'^4 - OP^4},$$

and this is true also in the limit; therefore

$$\frac{\text{moment of inertia of } OKP}{OP^4} = \frac{\tfrac{1}{2} OP^2 . \Delta POP'}{OP'^4 - OP^4}$$

$$= \frac{\tfrac{1}{4} OP^3 . PP' \sin\alpha}{4\,OP^3 . PP' \cos\alpha} = \tfrac{1}{16}\tan\alpha.$$

Therefore the moment of inertia of the area generated by the radius vector in describing any arc from the pole, about the pole is $\frac{1}{16}$ of the product of the fourth power of the extreme radius by the tangent of the angle of the spiral.

Obs. The same method may be applied to numerous other cases.

Prop. X. *In an equiangular spiral, the polar radii of points whose vectorial angles are in* A. P. *are themselves in* G. P.

Let $OP$, $OQ$, $OR$ (fig. 7) be three radii whose vectorial angles are in A. P., then the angles $POQ$, $QOR$ are equal. Therefore, since $PQKO$, $QRKO$ are similar figures and $OP$, $OQ$ corresponding lines, $OQ$, $OR$ are also corresponding lines; therefore

$$OP : OQ = OQ : OR,$$

and therefore $OP$, $OQ$, $OR$ are in G. P. But these are any three lines having their vectorial angles in A. P.; therefore, &c. Q.E.D.

Cor. I. If three points have their vectorial angles in A. P. the logarithms of the ratios of their polar radii to the prime radius are also in A. P.

Cor. II. If a series of points be taken whose vectorial angles are multiples of a given angle, the logarithms of the ratios of their radii to the prime radius will be the same multiples of the logarithm of the ratio of the radius corresponding to the given vectorial angle to the prime radius.

Cor. III. If a series of points be taken whose vectorial angles are *commensurable*, these angles will vary as the logarithms of the ratios of the corresponding radii vectores to the prime radius.

Obs. By a proof *ex absurdo* this can readily be extended to the case when the angles are *incommensurable;* hence, generally, *The logarithm of the ratio of the radius vector at any point of an equiangular spiral to the prime radius varies as the vectorial angle.*

Or, if $OA$ (fig. 8) be the prime radius, $OP$ any other radius, then

$$\log \frac{OP}{OA} = \mu . \angle POA.$$

TO DETERMINE $\mu$. Since $\mu$ has the same value wherever $P$ be taken, the above equation will hold when $P$ approaches indefinitely near to $A$. But in this case

$$\log \frac{OP}{OA} = \log \left(1 - \frac{AP \cos\alpha}{OA}\right) = -\frac{AP \cos\alpha}{OA} \text{ ultimately,}$$

and $$\angle POA = \frac{PA \sin\alpha}{OA} \text{ ultimately;}$$

hence $$-\frac{AP \cos\alpha}{OA} = \mu \frac{AP \sin\alpha}{OA},$$

$$\mu = -\cot\alpha.$$

Hence $P$ being any point on the curve,

$$\log \frac{OP}{OA} = -\cot\alpha . \angle POA,$$

or, with the notation of analytical geometry,

$$\log \frac{r}{a} = -\theta \cot\alpha.$$

If we measure $\theta$ in the direction of $r$ *increasing*, this becomes

$$\log \frac{r}{a} = \theta \cot\alpha,$$

or $$r = a\varepsilon^{\theta \cot\alpha},$$

the usual equation to the spiral.

W.

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# CONSERVATION OF FORCE.

By *J. M. Wilson, B.A.*, Fellow of St. John's College, Cambridge, and Natural Philosophy Master in Rugby School.

AMONG the mechanical laws established by philosophers of the last century, that of the conservation of force in certain systems of particles, and in machines moving under certain conditions, occupies a distinguished place. It was not only the final answer to the numerous attempts to produce perpetual motion, but was the germ of the far more splendid theorem which we owe to our own time and to philosophers still living. This theorem may be regarded by mathematicians, by men of science, and by practical men alike, as of the greatest importance, and it is probable that when a survey is taken of the progress of science during this century, our time will be honourably distinguished as having been first to grasp the theorem in all its generality.

In this *Journal* it will be most suitable to treat the subject as a mathematical one, and to illustrate it with one or two elementary examples; with a sketch of its mode of application to other branches of science not as yet completely under the domain of mathematics.

Consider a particle $m$ attracted to a fixed centre of force; let the centre of force be taken as the origin, $x$, $y$, $z$ the coordinates of $m$ referred to rectangular axes, $r^2 = x^2 + y^2 + z^2$, and $\phi(r)$ the law of attracting force.

Then if $v$ be the velocity at distance $r$, $V$ at distance $R$,

$$\tfrac{1}{2}mV^2 - \tfrac{1}{2}mv^2 = -\int_r^R \phi(r)\,dr.$$

We proceed to consider the significance of this equation.

The assumptions must be specially noticed; they are that the force acts in a straight line, and depends solely on the distance.

The left-hand side of the equation is the gain of *vis viva* (we define *vis viva* as $\tfrac{1}{2}mv^2$), and the right-hand side is the sum of the products of the moving force into the distance through which its point of application moves in the direction of the force; now this is the definition of work done (Griffin's *Dynamics of a Rigid Body*, Art. 160). Hence the increase of *vis viva* in the system as it moves from one position to another is equal to the work done.

In the simple case of a particle falling from rest under the action of gravity $\frac{1}{2}mv^2 = mgh$, or the *vis viva* is equal to the product of the force into the space through which it has acted.

Before passing to any more general case, it will be well to notice a difficulty which has been raised on this point by the most eminent of living philosophers in England, Faraday (*Researches in Chemistry and Physics*, p. 447. *Proceedings of Royal Institution*, Vol. II., p. 352). He says, "Assume two particles of matter, $A$ and $B$, in free space, "and a force in each or both by which they gravitate towards "each other, the force being unalterable for an unchanging "distance, but varying inversely as the square of the distance "when the latter varies. Then at the distance of 10, the "force may be estimated as 1; whilst at the distance of 1, "*i.e.* one-tenth of the former, the force will be 100: and if "we suppose an elastic spring to be introduced between the "two as a measure of the attractive force, the power com-"pressing it will be a hundred times as much in the latter "case as in the former. But from whence can this enormous "increase of power come? If we say that it is the character "of this force, and content ourselves with that as a sufficient "answer, then it appears to us we admit a *creation* of power, "and that to an enormous amount; yet by a change of con-"dition so small and simple, as to fail in leading the least "instructed mind to think that it can be a sufficient cause:— "we should admit a result which would equal the highest "acts our minds can appreciate of the working of infinite "power upon matter; we should let loose the highest law "in physical science which our faculties permit us to perceive, "namely, the conservation of force."......

"The principle of the conservation of force would lead "us to assume that when $A$ and $B$ attract each other less "because of increasing distance, then some other exer- "tion of power, either within or without them, is propor- "tionately growing up; and again, that when their distance "is diminished, as from 10 to 1, the power of attraction, "now increased a hundred fold, has been produced out of "some other form of power which has been equivalently "reduced."

...... "If we consider two particles of matter at a certain "distance apart, attracting each other under the power of "gravity and free to approach, they will approach; and "when at only half the distance, each will have had stored "up in it, because of its *inertia*, a certain amount of me- "chanical force. This must be due to the force exerted, "and if the conservation principle be true, must have con- "sumed an equivalent proportion of the cause of attraction; "and yet, according to the definition of gravity, the attrac- "tive force is not diminished thereby, but increased four- "fold, the force growing up within itself more rapidly, the "more it is occupied in producing other force."

These passages, and the lecture from which they are extracted, are well worthy of the most careful study. The meaning of the author is expressed with that perfect lucidity of style and illustration that it is scarcely possible to have a misconception of it. It contains, I believe, a great truth, and a most dangerous fallacy. I wish to be understood as approaching this subject with the greatest diffidence, and only from a profound belief that the subject is not brought so prominently before young students in mathematics as it deserves.

The truth in the above quotation is that it is very hard to conceive a force varying as the inverse square of the distance; to conceive how, one lone particle existing in space, the creation of a second could so affect it as to create a force of attraction which did not previously exist;—the fallacy is in supposing that this affects the doctrine of conservation of force at all.

In order to make this subject clearer, I shall introduce a new term, which I have found useful in verbally explaining this doctrine, and shall speak in future of *the conservation of potency*. I use this term because it can be applied without inaccuracy to express the capability of exercising force which a body possesses by virtue of its position, or its velocity, or its temperature, or the other qualities which must be known if we would have a complete physical description of the body. It is difficult to speak of the *force* of water in a milldam; and more difficult to compare the *forces* of the water in two cisterns at different elevations, but the meaning of the potency of the water is obvious, viz. the work which the water possesses the power of doing by falling from its present position to some lower level, to which reference is tacitly made.

Conceive two particles at a distance of one foot. The system possesses a certain amount of potency, viz. the work which would be done by the particles coming together under the influence of their own attractions. This we may call the *potency of position* of the system, and exists without increase or decrease, even if the particles are held apart by some statical pressure. This potency therefore depends solely on position and equals $-\int_0^r \phi(r)\,dr$ when the particles are at a distance $r$ from one another. Also since

$$-\int_r^R \phi(r)\,dr = -\int_0^R \phi(r)\,dr + \int_0^r \phi(r)\,dr,$$

we find that the loss of potency of position in a system is equal to the *vis viva* gained, and is therefore equal to the work done. Let us call this *vis viva the potency of velocity;* we then see that the loss of potency of position is balanced by the gain in potency of velocity; and therefore that the sum of the two potencies is constant. This is the doctrine of conservation of force in its simplest form.

The motion of a pendulum will form an admirable illustration. It hangs at rest: you raise it to one side, or communicate to it a potency of position at the expense of an equivalent muscular effort. It is allowed to swing: as it descends it loses potency of position, and gains potency of velocity, and when it reaches the lowest point of its arc of vibration, it has lost all the former, and gained its maximum of the latter. That these are equivalent is proved by the ascent of the pendulum on the opposite side. It rises to the same height, and loses all its velocity. Its potency of position has been restored at the expense of its potency of velocity. But such a pendulum can do no work. If it has to communicate potencies of motion to ill constructed supports, of heat to axles where there is friction, of motion to the air through which it swings, its own potencies suffer a diminution.

Take now Faraday's hypothesis. Let the two particles which were at the distance 10, and attracting one another with the force 1, approach to the distance 1, and attract with the force of 100. There is, it is argued, an enormous gain of power: the spring is more compressed than it was before. But this power is statical pressure, and has no connection with the notion of conservation of force; though from the use of the terms 'force of gravitation,' and 'conservation of force,' the confusion is not to be wondered at. The spring may remain compressed for ages. There is no work done; any more than by the huge boulders that lie on a hillside. Take the spring away, the particles fly together and do work: set the boulder in motion, and it will do work; they both have a potency of position. When it is said that the force is still exercised, and is a hundred-fold greater; the answer is that statically the force is or may be exerted, but dynamically it is non-existent: it may be a curious speculation, and a most important object to examine, what other effects besides pressure gravitation may produce (as heat or electricity) on bodies at rest; and the knowledge that it did produce other effects would modify the statement of the application of this doctrine to bodies in motion to some slight extent; by adding to the potencies of velocity and position some other potency as of heat which depended on the position or velocity or both, but not on the time explicitly; but such a discovery, though it would be of immense importance as correlating gravitation with other forces, would not make the statement I have been examining one whit the less fallacious. It is a legitimate object of inquiry what *else* the gravitating force is doing when a body is at rest besides producing pressure: and if it is doing anything, the residual action will be subject to the law of conservation of potency: but the pressure has no potency whatever.

Hence by the approach of two particles to one another under mutual attraction, while "mechanical force because of their inertia" is growing up, *i.e.* a potency of velocity is being gained, and the attractive force is also increased, a potency of position is lost.

We have seen then in a system of two particles that on the assumptions given above, the conservation of force or of potency holds. It is not difficult to extend this mathematically to any number of particles.

But I shall now proceed to the converse; viz. assuming it as axiomatic that force cannot be created, to deduce the axioms laid down above.

Since force, *i.e.* potency, cannot be created by any change in position of the particles of a system caused by their mutual action, the gain in potency of velocity must equal the loss in potency of position. For if it exceeded the loss, it might restore the potency of position *and* do work, contrary to the hypothesis. Now the potency of position is obviously a function of $r$ alone. Hence $\frac{1}{2}mv^2$,* (the inertia being unaffected) is also a function of $r$ alone.

And if $F$ be the force at distance $R$, $f$ at $r$, and since by our axiom

$$\tfrac{1}{2}mV^2 - \tfrac{1}{2}mv^2 = -\int_r^R F dr,$$

and the left-hand side is a function of $r$ alone, therefore its differential or $F$ involves $r$ alone.

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* While the inertia remains unaffected no collateral force can spring up: if the doctrine of conservation holds true.

Also, if we suppose the particle to move so as to come to the surface of a sphere concentric with the origin of the force, whose radius is $r$, in motion on that surface $d(v^2)=0$, therefore

$$Xdx+Ydy+Zdz=0,$$

where $X$, $Y$, $Z$ are the components parallel to the axes of $F(r)$.

Also $xdx+ydy+zdz=\frac{1}{2}d(r^2)=0$ for such motion, therefore

$$\frac{X}{x}=\frac{Y}{y}=\frac{Z}{z},$$

or the force tends to the origin. Hence these assumptions are convertible.

I shall now proceed to illustrate the doctrine by discussing the motion of a mechanical contrivance familiar to all, viz. the crank. I select it, not only from its intrinsic importance, as the ordinary means for converting oscillating rectilinear motion into continuous rotatory, but also from a lively recollection of a discussion that our scholars' table once witnessed, in which the most elementary principles of conservation of force were ignored, and the commonest fallacies vigorously defended by nearly every one who took part in the discussion; I believe, further, that it is not discussed in any text-book used in Cambridge.

Let $DF$ (fig. 9) be the piston of an engine, which moves capable of exercising a uniform pressure, from $E$ to $B$ and back. $AC$ the arm of the crank $=r$, $CD$ the connecting rod of length $a$.

And at the time $t$, let

$$DE=y,\ AC=r,\ CD=a,\ \angle CDA=\phi,\ CAD=\theta.$$

We have then the following equations by geometry:

$$r\sin\theta=a\sin\phi \ \dots\dots\dots\dots\dots\dots \text{(i)},$$

$$y+r\cos\theta+a\cos\phi=r+a \ \dots\dots\dots\dots \text{(ii)}.$$

Differentiate with respect to $t$, and eliminate $\frac{d\phi}{dt}$,

and $$\frac{dy}{dt}=r\frac{\sin(\theta+\phi)}{\cos\phi}\frac{d\theta}{dt} \ \dots\dots\dots\dots\dots \text{(iii)},$$

also $P$ being the efficient force at $D$, $P'$ at $C$ perpendicular to the arm of the crank, $Q$ the pressure along $CD$ which is the resultant of $P$ and a force transverse to $P$, then

$$P = Q \cos\phi,$$

$$P' = Q \sin(\theta + \phi),$$

and therefore $$P' = P \sec\phi \sin(\theta + \phi) \text{ .............. (iv)},$$

and therefore $$P\frac{dy}{dt} = P'r\frac{d\theta}{dt} \text{ .................... (v)},$$

that is, the work done at $D$ is equal to the work done at $C$, or no work is lost by pressure or otherwise. If we take the average of a whole revolution, the velocities of $D$ and $C$ are as $2 : \pi$, and therefore

$$P' : P :: 2 : \pi;$$

therefore $$P' = \tfrac{7}{11}P \text{ nearly},$$

therefore the work done by the crank is the same as if it were turned by a uniform force acting on the arm of the crank at a distance from its center $= \frac{7}{11}$th of the length of the arm.

The velocity of $C$ will vary, for if $R$ is the resistance of the machine, *i.e.* a measure of the work done, the friction, &c., supposed applied at $C$ perpendicular to the arm, and of uniform amount, $m$ the mass of the crank

$$m\frac{d^2\theta}{dt^2} = r(P' - R);$$

therefore $$2m\frac{d^2\theta}{dt}\frac{d\theta}{dt} = 2P'r\frac{d\theta}{dt} - 2Rr\frac{d\theta}{dt},$$

$$= 2P\frac{dy}{dt} - 2Rr\frac{d\theta}{dt} \text{ by (v)};$$

therefore $$m\left(\frac{d\theta}{dt}\right)^2 = C + 2Py - 2Rr\theta.$$

Let $\omega$ be the angular velocity at the position of the crank when $D$ is at $E$, then

$$C = m\omega^2,$$

and the angular velocity at any point is given by the equation

$$m\omega'^2 = m\omega^2 + 2Py - 2Rr\theta,$$

if the engine is double acting, and the motion is steady in the long run, *i.e.* is not gradually decreasing or increasing, the velocities at opposite positions of the crank must be the same, therefore

$$2P(2r) - 2Rr\pi = 0;$$

therefore $$R = \frac{2}{\pi}P = \tfrac{7}{11}P \text{ as before.}$$

Hence, the product of $R$ into the space through which it moves in a semi-revolution = product of $P$ into space through which it moves, *i.e.* no work is lost. This principle is universally applicable to combinations of the mechanical powers.

In fact we have by the principle of virtual velocities

$$\Sigma\{Pdx\} = 0,$$

where $P$ represents any force, $dx$ a hypothetical motion of its point of application in direction of the force: and this equation comprehends the whole of Statics.

The integral of this equation

$$\int \Sigma(Pdx) = C$$

comprehends the whole of Dynamics.

(*To be continued.*)

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# NOTE.

THE following geometrical proof of the theorem proved analytically in Todhunter's *Integral Calculus*, Art. 90, has never, we think, been printed, and will probably be new to many of our readers.

Let $PP' = \delta s$ (fig. 10) be any small arc of a curve; $PY$, $P'Y'$ tangents meeting in $T$; $OY$, $OY'$ perpendiculars from the origin on the tangents; and let $OY$, $TY'$ meet in $H$. Then ultimately, when $P'$ coincides with $P$, the angles at $H$ are right angles. Let

$OY = p$, $OY' = p + \delta p$, $PY = u$, $PY' = u + \delta u$, $\angle YOY' = \delta\theta$;

then $$\delta u + \delta s = P'Y' - PY + PP' = PY' - PY$$

$$= HY' = OY'\,\delta\theta \text{ ult.};$$

therefore $$\frac{du}{d\theta} + \frac{ds}{d\theta} = p,$$

or $$u + s = \int p\,d\theta \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

Again, $$-\delta p = YH = TY\,\delta\theta \text{ ult.};$$

therefore $$\frac{dp}{d\theta} = -u \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}.$$

From these results we may readily deduce the formula $\rho = \frac{rdr}{dp}$, thus

$$\rho = \frac{ds}{d\theta} = p - \frac{du}{d\theta} \quad \text{by (i)}$$

$$= p - \frac{du}{dp}\frac{dp}{d\theta}$$

$$= p + u\frac{du}{dp} \quad \text{by (ii)}$$

$$= \frac{rdr}{dp},$$

since $r^2 = p^2 + u^2$, and therefore $rdr = pdp + udu$.

## QUERY.

Can any correspondent give a satisfactory explanation of the following phenomenon?

The sun is shining through a window of a light room and illuminates the opposite wall, the shadows of the window bars being distinctly marked. A cloud comes over the sun and destroys the illuminated patch, but while vanishing the patch is observed to shift along the wall for a considerable distance in the direction opposite to that in which the cloud is moving. This displacement of the patch of light is the phenomenon in question.

It has been suggested that the apparent shadow of any point, being generally the geometrical shadow due to the centre of the sun, will, when the disc is partly covered, be the geometrical shadow due to the approximate centre of the uncovered part. Now when a cloud is coming over the sun this centre will move before the cloud from the centre of the complete disc to the circumference, and hence we may expect a corresponding shifting of the shadows on the wall.

But against this explanation we must urge that, estimating the sun's diameter roughly at half-a-degree, the whole distance through which the shadow-centre can be displaced is a quarter-of-a-degree, and therefore the whole displacement of any point of the shadow of the window on the wall should subtend at the window an angle not exceeding a quarter-of-a-degree: but the observed displacement is roughly estimated at about three times this. Hence the explanation only accounts for a small part of the observed displacement.

Another suggestion is, that the diffraction of the rays at the edge of the cloud give rise to the observed phenomenon, but it does not seem probable that the clouds can present an edge capable of causing such diffraction; thus we are obliged to abandon this explanation also.

G. G.

# ON DETERMINANTS.*

## A CHAPTER OF ELEMENTARY ALGEBRA.

*By P. G. Tait, M.A.*

THE following article was written some years ago for the use of the Mathematical Classes in Queen's College, Belfast, and would have been published in a separate form if I had still been connected with that institution. A good deal of the necessity for such a paper has been removed, by the recent publication of Mr. Ferrers' *Trilinear Coordinates*, in which there is an excellent elementary chapter on the subject. But as that volume is not for the reading of the bulk of University men, and as Determinants are only treated in it so far as they are required for the special objects of the Treatise, I have thought that the publication of the present paper might still be to some extent useful. I may merely premise, that as my object is explanation, I have studiously avoided novelty and originality.

I. If we have two equations of the form

$$\left.\begin{array}{l} ax+by\ =c \\ a_1x+b_1y=c_1 \end{array}\right\} \text{.....................} \quad (1),$$

containing two unknown quantities $x$ and $y$, we find, on eliminating $y$

$$(ab_1-ba_1)\,x=(cb_1-bc_1) \text{ ............ } (2).$$

Had we eliminated $x$, we should have had

$$(ab_1-ba_1)\,y=(ac_1-ca_1) \text{ ............ } (2').$$

It is to be observed, that the coefficient of $x$ or $y$ in the resulting equation is $ab_1-ba_1$. We may write this $\begin{vmatrix} a, & b \\ a_1, & b_1 \end{vmatrix}$

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* Since the above was received, Mr. Todhunter's *Theory of Equations* has been published; it contains some chapters on Determinants, and the method of treating the subject therein adopted is somewhat similar to that of the present paper. But it has seemed advisable nevertheless to publish this, as it is thought that many students who will not read Theory of Equations at all, will be glad to have their "Elementary Algebra" extended by the short and simple explanation of the Theory of Determinants here offered to them.—EDD.

if we agree that the latter form shall indicate partial multiplication along the diagonals of the square, taking the difference of the products, and treating that in the right-hand descending diagonal as positive.

It is also to be observed, that if $c = c_1 = 0$, equations (1) are not necessarily consistent. In fact, they are only so, if $-\frac{y}{x} = \frac{a}{b} = \frac{a_1}{b_1}$, or if $ab_1 - ba_1 = 0$. Hence the expression $ab_1 - ba_1 = \begin{vmatrix} a, & b \\ a_1, & b_1 \end{vmatrix}$ may be described, either as the coefficient of one of the two unknown quantities in a pair of equations of the first degree, when the other is eliminated; or as the quantity which, when equated to zero, gives the condition of the simultaneous existence of two *homogeneous* equations of the first degree between two unknown quantities.

This expression is called a Determinant of the *second* order. It may be well to observe, that since $\begin{vmatrix} a, & b \\ a_1, & b_1 \end{vmatrix} = ab_1 - ba_1$, we have

$$(\alpha) \qquad \begin{vmatrix} a, & b \\ a_1, & b_1 \end{vmatrix} = \begin{vmatrix} a, & a_1 \\ b, & b_1 \end{vmatrix},$$

$$(\beta) \qquad \begin{vmatrix} a, & b \\ a, & b \end{vmatrix} = \begin{vmatrix} a, & a \\ b, & b \end{vmatrix} = 0,$$

$$(\gamma) \qquad \begin{vmatrix} a, & a_1 \\ b, & b_1 \end{vmatrix} = -\begin{vmatrix} a_1, & a \\ b_1, & b \end{vmatrix} = -\begin{vmatrix} b, & b_1 \\ a, & a_1 \end{vmatrix},$$

$$(\delta) \qquad \begin{vmatrix} \mu a, & \mu a_1 \\ b, & b_1 \end{vmatrix} = \begin{vmatrix} \mu a, & a_1 \\ \mu b, & b_1 \end{vmatrix} = \mu \begin{vmatrix} a, & a_1 \\ b, & b_1 \end{vmatrix},$$

or, in other words,

($\alpha$) In a determinant of the second order, *columns* may be changed into *rows*, and *vice versâ.*

($\beta$) If the columns or the rows be alike, the determinant vanishes.

($\gamma$) If the columns or rows be interchanged, the *sign* of the determinant is changed.

(δ) If a row or column be multiplied by any quantity, the whole determinant is so.

It is evident that (β) may be at once proved from (γ). This will be again referred to. To solve then the pair of equations

$$ax + by = c,$$
$$a_1x + b_1y = c_1,$$

we may evidently write (and indeed it is only the well-known method of cross multiplication in another form)

$$\begin{vmatrix} a, & b \\ a_1, & b_1 \end{vmatrix} x + \begin{vmatrix} b, & b \\ b_1, & b_1 \end{vmatrix} y = \begin{vmatrix} c, & b \\ c_1, & b_1 \end{vmatrix},$$

where the multiplier of $y$ vanishes by (β),

or
$$\begin{vmatrix} a, & a \\ a_1, & a_1 \end{vmatrix} x + \begin{vmatrix} b, & a \\ b_1, & a_1 \end{vmatrix} y = \begin{vmatrix} c, & a \\ c_1, & a_1 \end{vmatrix},$$

where the multiplier of $x$ vanishes as before. Either of the two unknown quantities is thus given as the ratio of two determinants.

II. Let us now consider the three homogeneous equations of the first degree,

$$\left.\begin{aligned} ax + by + cz &= 0 \\ a_1x + b_1y + c_1z &= 0 \\ a_2x + b_2y + c_2z &= 0 \end{aligned}\right\} \text{..................} \quad (3).$$

What is the condition of their simultaneous existence?

Eliminating separately $y$ and $z$ from the last two of equations (3), by I. (β), we have

$$\begin{vmatrix} a_1, & b_1 \\ a_2, & b_2 \end{vmatrix} x + \begin{vmatrix} c_1, & b_1 \\ c_2, & b_2 \end{vmatrix} z = 0,$$

$$\begin{vmatrix} a_1, & c_1 \\ a_2, & c_2 \end{vmatrix} x + \begin{vmatrix} b_1, & c_1 \\ b_2, & c_2 \end{vmatrix} y = 0,$$

or by I. (γ),

$$\frac{x}{\begin{vmatrix} b_1, & c_1 \\ b_2, & c_2 \end{vmatrix}} = \frac{y}{\begin{vmatrix} c_1, & a_1 \\ c_2, & a_2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_1, & b_1 \\ a_2, & b_2 \end{vmatrix}}.$$

Dividing in order the terms of the first equation in (3) by these three equal quantities, it becomes

$$a\begin{vmatrix} b_1, & c_1 \\ b_2, & c_2 \end{vmatrix} + b\begin{vmatrix} c_1, & a_1 \\ c_2, & a_2 \end{vmatrix} + c\begin{vmatrix} a_1, & b_1 \\ a_2, & b_2 \end{vmatrix} = 0 \text{.......} \quad (4),$$

which is the required condition.

We may write the left-hand member as follows:

$$\begin{vmatrix} a, & b, & c \\ a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \end{vmatrix} \text{........................} (4'),$$

and the comparison of these expressions will give us the interpretation of the latter, which is called a determinant of the *third* order.

III. But there is another method of finding the condition that equations (3) may be consistent. Multiply the second by $\lambda$, the third by $\mu$, and add; then, since $\lambda$ and $\mu$ may be any quantities whatever, we may take them such as to make the coefficients of two of the unknowns vanish; whence, if the equations be simultaneously true, that of the third must vanish; hence

$$\left.\begin{aligned} a + a_1\lambda + a_2\mu = 0 \\ b + b_1\lambda + b_2\mu = 0 \\ c + c_1\lambda + c_2\mu = 0 \end{aligned}\right\} \text{..................} \quad (5),$$

or, solving by I. the last two for $\lambda$ and $\mu$,

$$\begin{vmatrix} b, & b_2 \\ c, & c_2 \end{vmatrix} + \begin{vmatrix} b_1, & b_2 \\ c_1, & c_2 \end{vmatrix} \lambda = 0,$$

$$\begin{vmatrix} b, & b_1 \\ c, & c_1 \end{vmatrix} + \begin{vmatrix} b_2, & b_1 \\ c_2, & c_1 \end{vmatrix} \mu = 0,$$

and substituting in the first equation of (5),

$$a\begin{vmatrix} b_1, & b_2 \\ c_1, & c_2 \end{vmatrix} + a_1\begin{vmatrix} b_2, & b \\ c_2, & c \end{vmatrix} + a_2\begin{vmatrix} b, & b_1 \\ c, & c_1 \end{vmatrix} = 0 \text{ ......} \quad (6),$$

which must therefore be the same as (4), as indeed is evident on multiplying out. Hence, referring to (4′), (4), and (6), we see that

$$\begin{vmatrix} a, & b, & c \\ a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \end{vmatrix} = \begin{vmatrix} a, & a_1, & a_2 \\ b, & b_1, & b_2 \\ c, & c_1 & c_2 \end{vmatrix} \ldots\ldots\ldots\ldots\ldots(7).$$

It is evident by (4) or (6) that if $a=a_1$, $b=b_1$, $c=c_1$, or $a=b$, $a_1=b_1$, $a_2=b_2$, &c., *i.e.*, if the numbers in any column or row be equal (in order) to those in any other column or row, the determinant vanishes of itself. Hence, let us take the complete equations with three unknowns,

$$\left.\begin{aligned} ax+by+cz&=d \\ a_1x+b_1y+c_1z&=d_1 \\ a_2x+b_2y+c_2z&=d_2 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(8).$$

Multiply the first by $\begin{vmatrix} b_1, & c_1 \\ b_2, & c_2 \end{vmatrix}$,

............... second by $\begin{vmatrix} b_2, & c_2 \\ b, & c \end{vmatrix}$,

............... third by $\begin{vmatrix} b, & c \\ b_1, & c_1 \end{vmatrix}$,

and add, we have

$$\begin{vmatrix} a, & a_1, & a_2 \\ b, & b_1, & b_2 \\ c, & c_1, & c_2 \end{vmatrix} x + \begin{vmatrix} b, & b_1, & b_2 \\ b, & b_1, & b_2 \\ c, & c_1, & c_2 \end{vmatrix} y + \begin{vmatrix} c, & c_1, & c_2 \\ b, & b_1, & b_2 \\ c, & c_1, & c_2 \end{vmatrix} z = \begin{vmatrix} d, & d_1, & d_2 \\ b, & b_1, & b_2 \\ c, & c_1, & c_2 \end{vmatrix},$$

which former results reduce at once to

$$\begin{vmatrix} a, & a_1, & a_2 \\ b, & b_1, & b_2 \\ c, & c_1, & c_2 \end{vmatrix} x = \begin{vmatrix} d, & d_1, & d_2 \\ b, & b_1, & b_2 \\ c, & c_1, & c_2 \end{vmatrix},$$

or, as previously shewn,

$$\begin{vmatrix} a, & b, & c \\ a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \end{vmatrix} x = \begin{vmatrix} d, & b, & c \\ d_1, & b_1, & c_1 \\ d_2, & b_2, & c_2 \end{vmatrix}.$$

This, by the help of the expressions (4) or (6) for the determinants, gives us at once the solution of three equations with three unknown quantities.

Also by (4) or (6) we see that if each number of a row or a column of a determinant of the third order be multiplied by the same quantity, the whole is multiplied by that quantity.

It is easy to see also that if two rows or two columns be interchanged the sign of the whole is changed. This will be evident, if we examine the corresponding change in expressions (4) or (6) and refer to I. ($\gamma$). Whence, if two rows or columns be identical, the whole is zero, as has been otherwise proved.

IV. It will be easy for any one who has followed us thus far to deduce the solution of the four equations of the first degree with four unknowns,

$$\left.\begin{array}{l} ax + by + cz + du = e \\ a_1x + b_1y + c_1z + d_1u = e_1 \\ a_2x + b_2y + c_2z + d_2u = e_2 \\ a_3x + b_3y + c_3z + d_3u = e_3 \end{array}\right\} \ldots\ldots\ldots\ldots\ldots (9),$$

in the form

$$\begin{vmatrix} a, & b, & c, & d \\ a_1, & b_1, & c_1, & d_1 \\ a_2, & b_2, & c_2, & d_2 \\ a_3, & b_3, & c_3, & d_3 \end{vmatrix} x = \begin{vmatrix} e, & b, & c, & d \\ e_1, & b_1, & c_1, & d_1 \\ e_2, & b_2, & c_2, & d_2 \\ e_3, & b_3, & c_3, & d_3 \end{vmatrix} \&\text{c.},$$

where

$$\begin{vmatrix} a, & b, & c, & d \\ a_1, & b_1, & c_1, & d_1 \\ a_2, & b_2, & c_2, & d_2 \\ a_3, & b_3, & c_3, & d_3 \end{vmatrix} = a \begin{vmatrix} b_1, & c_1, & d_1 \\ b_2, & c_2, & d_2 \\ b_3, & c_3, & d_3 \end{vmatrix} - b \begin{vmatrix} c_1, & d_1, & a_1 \\ c_2, & d_2, & a_2 \\ c_3, & d_3, & a_3 \end{vmatrix}$$

$$+ c \begin{vmatrix} d_1, & a_1, & b_1 \\ d_2, & a_2, & b_2 \\ d_3, & a_3, & b_3 \end{vmatrix} - d \begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix}.$$

Whence, as before, it may be seen that *if the numbers in a row or column of the determinant be multiplied by any quantity, the whole is so multiplied.*

*If two columns or rows be interchanged, the sign of the whole is changed.*

*If two be equal the value is zero.*

And generalizing, the condition of simultaneous existence of $n$ homogeneous equations of the first degree among $n$ unknowns is that the determinant

$$\begin{vmatrix} a, & b, & c, & \dots\dots\dots \\ a_1, & b_1, & c_1, & \dots\dots\dots \\ a_2, & b_2, & c_2, & \dots\dots\dots \\ \dots & \dots & \dots & \dots\dots\dots \\ \dots & \dots & \dots & \dots\dots\dots \end{vmatrix}$$

(formed of their coefficients) be equal to zero; and if there be constant terms in the equations, and $(n-1)$ of the variables be eliminated, the coefficient of the $n^{\text{th}}$ is the above determinant; the other term of the final equation is the same determinant with the constant terms in the $n$ equations substituted in order for the coefficients of the $n^{\text{th}}$ unknown.

It is easy to see also that *a determinant of any order may be expressed as a homogeneous function of the first degree of the constituent numbers of any column or row, the terms being alternately + and − when the determinant is of an even order, and all + when it is of an odd order.*

Whence it follows that *when one column or row is multiplied by any quantity the whole is so.*

Whence also it follows that *if the numbers in any row or column be broken up into sums of two or more quantities (including zero), the whole determinant may be broken up into the sum of two or more.*

Thus

$$\begin{vmatrix} a, & b+\beta, & c \\ a_1, & b_1+\beta_1, & c_1 \\ a_2, & b_2+\beta_2, & c_2 \end{vmatrix}$$

$$= \begin{vmatrix} a, & b, & c \\ a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \end{vmatrix} + \begin{vmatrix} a, & \beta, & c \\ a_1, & \beta_1, & c_1 \\ a_2, & \beta_2, & c_2 \end{vmatrix},$$

and so on.

Also it may be seen that *on interchanging two columns or two rows, the sign only is changed.* Hence, *if two rows or columns be identical, the whole vanishes.*

Combining the last three theorems, we see that *the value of a determinant is unchanged if the numbers in any row or column, multiplied all by the same quantity, be added to, or subtracted from, the corresponding ones in another row or column, the first-mentioned row or column being left unaltered.* This last proposition is often of very great use in the simplification of complex systems of equations.

V. The usual method of stating the foregoing propositions is somewhat more difficult for the beginner, it may, however, be put as follows; and it is, as a general statement, in reality far less troublesome than the excessively elementary process which we have given above.

The method is (briefly) this. Having $n^2$ quantities

$$\begin{gathered}
{}^1a_1,\ {}^1a_2,\ {}^1a_3,\ \ldots {}^1a_n, \\
{}^2a_1,\ {}^2a_2,\ \ldots\ldots\ldots {}^2a_n, \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
{}^na_1,\ {}^na_2,\ \ldots\ldots\ldots {}^na_n,
\end{gathered}$$

let us form every possible product of the form

$${}^1a_1 . {}^2a_2 . {}^3a_3 . {}^4a_4 \ldots\ldots\ldots\ldots {}^na_n,$$

which can be written by interchanging two indices *or* two suffixes, changing or not, according to a certain law, the sign of the whole for each such interchange, then the sum of all such with their proper signs is the determinant formed with the given quantities, the indices referring to rows and the suffixes to columns. Representing it as

$$\Sigma \pm {}^1a_1 . {}^2a_2 . {}^3a_3 \ldots\ldots\ldots {}^na_n,$$

it is easy to prove generally the above properties, and thence to deduce the application of determinants to the solution of linear equations. For farther information we refer to the treatise of Brioschi (translated into French by Combescure) or to that of Spottiswoode.*

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* Or to Todhunter's *Theory of Equations*, where the manner of treating the subject is very thoroughly explained.—EDD.

VI. Examples.

$$\begin{vmatrix} a, & a^2, & a^3 \\ b, & b^2, & b^3 \\ c, & c^2, & c^3 \end{vmatrix} = abc \begin{vmatrix} 1, & a, & a^2 \\ 1, & b, & b^2 \\ 1, & c, & c^2 \end{vmatrix}.$$

And the factor $\begin{vmatrix} 1, & a, & a^2 \\ 1, & b, & b^2 \\ 1, & c, & c^2 \end{vmatrix}$ vanishes if $a=b$, or $b=c$, or $c=a$.

Its value is therefore evidently

$$(a-b)(b-c)(c-a).$$

Again, $\begin{vmatrix} 1, & a^2, & a^3 \\ 1, & b^2, & b^3 \\ 1, & c^2, & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$,

and $\begin{vmatrix} 1, & a, & a^3 \\ 1, & b, & b^3 \\ 1, & c, & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$.

Hence, to find the relations between the roots and coefficients of a cubic equation, let them be $a, b, c$ and $x, y, z$, we have

$$\left.\begin{array}{l} a^3+xa^2+ya+z=0 \\ b^3+xb^2+yb+z=0 \\ c^3+xc^2+yc+z=0 \end{array}\right\};$$

therefore $\begin{vmatrix} a^3, & a^2, & a \\ b^3, & b^2, & b \\ c^3, & c^2, & c \end{vmatrix} + \begin{vmatrix} 1, & a^2, & a \\ 1, & b^2, & b \\ 1, & c^2, & c \end{vmatrix} z = 0$,

or $$z=-abc.$$

$$\begin{vmatrix} a^3, & a^2, & 1 \\ b^3, & b^2, & 1 \\ c^3, & c^2, & 1 \end{vmatrix} + y \begin{vmatrix} a, & a^2, & 1 \\ b, & b^2, & 1 \\ c, & c^2, & 1 \end{vmatrix} = 0,$$

or $$y=ab+bc+ca.$$

Similarly $$x=-(a+b+c).$$

Again, solve the equations

$$13x+25y+40z=27,$$
$$9x+18y+30z=19,$$
$$4x+9y+13z=10.$$

We get at once, solving for $y$, suppose,

$$\begin{vmatrix} 25, & 40, & 13 \\ 18, & 30, & 9 \\ 9, & 13, & 4 \end{vmatrix} y = \begin{vmatrix} 27, & 40, & 13 \\ 19, & 30, & 9 \\ 10, & 13, & 4 \end{vmatrix},$$

or

$$\begin{vmatrix} -1, & 1, & 13 \\ 0, & 3, & 9 \\ 1, & 1, & 4 \end{vmatrix} y = \begin{vmatrix} 1, & 1, & 13 \\ 1, & 3, & 9 \\ 2, & 1, & 4 \end{vmatrix},$$

or

$$\begin{vmatrix} -1, & 1, & 10 \\ 0, & 3, & 0 \\ 1, & 1, & 1 \end{vmatrix} y = \begin{vmatrix} 1, & 1, & 10 \\ 1, & 3, & 0 \\ 2, & 1, & 1 \end{vmatrix}.$$

Hence

$$(-3-30)\,y = (3-1-50),$$

$$33y = 48.$$

And so on. We have given the reductions in full, but a little practice will enable the learner almost completely to dispense with them. As another example take the determinant

$$\begin{vmatrix} x, & y, & z \\ z, & x, & y \\ y, & z, & x \end{vmatrix}.$$

It is evidently equal to

$$\begin{vmatrix} x+\lambda z+\mu y, & y+\lambda x+\mu z, & z+\lambda y+\mu x \\ z, & x, & y \\ y, & z, & x \end{vmatrix},$$

whatever be $\lambda$ and $\mu$, and is therefore divisible by $x+\mu y+\lambda z$ if the numbers in the first row are so, *i.e.* if $\mu = \frac{1}{\lambda} = \frac{\lambda}{\mu}$,

that is $\mu^2 = \frac{1}{\mu}$, or $\mu^3 = 1$, while $\lambda = \mu^2$.

Hence, if

$$i^3 = 1,$$

$$x + iy + i^2 z$$

is a factor, or the determinant can be broken into three linear factors, whence the theorem

$$x^3+y^3+z^3-3xyz = (x+y+z)(x+iy+i^2z)(x+jy+j^2z),$$

where $i$ and $j$ are imaginary cube roots of unity.

It is easy to see that a similar theorem holds for determinants of any order if composed like that just treated.

Finally, let us consider

$$\begin{vmatrix} 1, & 1, & 1 \\ 1, & 1+\alpha, & 1 \\ 1, & 1, & 1+\beta \end{vmatrix}.$$

This vanishes by two columns becoming identical if $\alpha=0$, or if $\beta=0$, whence its value is easily seen to be $\alpha\beta$. And the same may be easily extended to higher orders.

VII. To multiply two determinants together, and to prove that the product is a determinant. Let us have two systems of any number of equations (say three, the proof is evidently quite general) such as

$$\left.\begin{aligned} ax + by + cz &= \xi \\ a_1x + b_1y + c_1z &= \eta \\ a_2x + b_2y + c_2z &= \zeta \end{aligned}\right\},$$

and

$$\left.\begin{aligned} \alpha\xi + \beta\eta + \gamma\zeta &= 0 \\ \alpha_1\xi + \beta_1\eta + \gamma_1\zeta &= 0 \\ \alpha_2\xi + \beta_2\eta + \gamma_2\zeta &= 0 \end{aligned}\right\}.$$

From these we may evidently eliminate $x$, $y$, $z$, $\xi$, $\eta$, and $\zeta$. Two methods suggest themselves. First, substitute for $\xi$, $\eta$, $\zeta$ in the second system from the first, and we have, collecting coefficients of $x$, $y$, $z$,

$$\begin{aligned} (\alpha a + \beta a_1 + \gamma a_2)\,x + \ldots\ldots\ldots\ldots\ldots &= 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots &= 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots &= 0, \end{aligned}$$

whence the condition of consistency is

$$\begin{vmatrix} \alpha a + \beta a_1 + \gamma a_2, & \&c., & \&c. \\ \&c., & \&c., & \&c. \\ \&c., & \&c., & \&c. \end{vmatrix} = 0 \ldots\ldots\ldots (10).$$

Secondly, we might write the equations thus:

$$\begin{array}{llll} ax + by + cz - \xi & & & = 0, \\ a_1x + b_1y + c_1z & - \eta & & = 0, \\ a_2x + b_2y + c_2z & & - \zeta & = 0, \\ & \alpha\xi + \beta\eta + \gamma\zeta & & = 0, \\ & \alpha_1\xi + \beta_1\eta + \gamma_1\zeta & & = 0, \\ & \alpha_2\xi + \beta_2\eta + \gamma_2\zeta & & = 0, \end{array}$$

whence the condition of consistency is, at once,

$$\begin{vmatrix} a, & b, & c, & -1, & 0, & 0 \\ a_1, & b_1, & c_1, & 0, & -1, & 0 \\ a_2, & b_2, & c_2, & 0, & 0, & -1 \\ 0, & 0, & 0, & \alpha, & \beta, & \gamma \\ 0, & 0, & 0, & \alpha_1, & \beta_1, & \gamma_1 \\ 0, & 0, & 0, & \alpha_2, & \beta_2, & \gamma_2 \end{vmatrix} = 0,$$

or, as may easily be seen by what precedes,

$$\begin{vmatrix} a, & b, & c \\ a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \end{vmatrix} \times \begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha_1, & \beta_1, & \gamma_1 \\ \alpha_2, & \beta_2, & \gamma_2 \end{vmatrix} = 0.$$

This must be the same condition as that in (10), and therefore the left-hand members can only differ by a numerical factor, and that we at once see must be unity. Hence, *to multiply two determinants of the same order, form another determinant of that order, the constituents of whose first row are the sums of the products of the numbers in the first row of one of the given determinants by those in the several columns of the other, and so on.*

If the determinants be not of the same order, increase the order of the lower by the following evidently lawful process

$$\begin{vmatrix} \alpha, & \beta \\ \gamma, & \delta \end{vmatrix} = \begin{vmatrix} 1, & 0, & 0 \\ 0, & \alpha, & \beta \\ 0, & \gamma, & \delta \end{vmatrix} = \begin{vmatrix} 1, & 0, & 0, & 0 \\ 0, & 1, & 0, & 0 \\ 0, & 0, & \alpha, & \beta \\ 0, & 0, & \gamma, & \delta \end{vmatrix},$$

and so on.

Ex. $$\begin{vmatrix} a, & b \\ c, & d \end{vmatrix} \begin{vmatrix} a, & b \\ c, & d \end{vmatrix} = \begin{vmatrix} a^2+bc, & ab+bd \\ ca+cd, & cb+d^2 \end{vmatrix},$$

or $$(ad-bc)^2 = (a^2+bc)(bc+a^2) - bc(a+d)^2,$$

a result easily verified.

Ex. Let $l, m, n, l_1, m_1, n_1, l_2, m_2, n_2$ be the direction-cosines of one set of rectangular axes with regard to another set, then

$$l^2+m^2+n^2=1,$$
$$\&c. \quad = \&c.,$$
$$ll_1+mm_1+nn_1=0,$$
$$\&c. \quad = \&c.$$

Now $$\begin{vmatrix} l, & m, & n \\ l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \end{vmatrix}^2 = \begin{vmatrix} l^2+m^2+n^2, & ll_1+mm_1+nn_1, & ll_2+mm_2+nn_2 \\ \&c., & \&c., & \&c. \\ \&c., & \&c., & \&c. \end{vmatrix}$$

$$= \begin{vmatrix} 1, & 0, & 0 \\ 0, & 1, & 0 \\ 0, & 0, & 1 \end{vmatrix} = 1.$$

Hence, evidently by (4),

$$l = \pm \begin{vmatrix} m_1, & n_1 \\ m_2, & n_2 \end{vmatrix}, \text{ \&c. as is well known.}$$

For the higher developments of this theory, and their applications to Analytical Geometry, the Differential Equations of Mechanics, &c., reference must be made to the writings of Jacobi, Cayley, Sylvester, Hermite, Salmon, Boole, and Pierce.

# THE RECTANGULAR HYPERBOLA.

By *W. G. Adams, B.A.*, Scholar of St. John's College, Cambridge, and Assistant Master at Marlborough College.

I. If $D$ and $E$ (fig. 11) be the feet of perpendiculars drawn from $Q$ and $P$ on the opposite sides of the triangle $PQR$, and $V$, $U$, $W$ the middle points of $PQ$, $QR$, $RP$ respectively; then $VD = VE = VP$.

For if a circle be described on $PQ$ as diameter, it will pass through the feet of the perpendiculars because the angles $PEQ$ and $QDP$ are right angles. Therefore

$$VD = VE = VP = \tfrac{1}{2}PQ.$$

Similarly $$WE = \tfrac{1}{2}PR.$$

Therefore the triangle $VEW$ is similar to the triangle $VUW$ and equal to it. Therefore the circle passing through the middle points of the sides of a triangle passes also through the feet of the perpendiculars let fall from the angular points on the opposite sides.

II. The locus of the centre of a rectangular hyperbola described about a triangle is the circle passing through the middle points of the sides of *the triangle*.

Let $PQR$ (fig. 12) be the triangle, and $U$, $W$, $V$ the middle points of its sides. Let $C$ be the centre of the hyperbola. Join $CU$, $CW$, and $CV$. Then since in the rectangular hyperbola conjugate diameters are equally inclined to the asymptotes, the angle between two diameters equals the angle between the chords which they bisect.

Hence the angle $VCW$ = angle $VPW$ or its supplement.

Therefore the angle $VCW$ is constant.

Similarly, the angle $VCU$ is constant and equal to the angle at $Q$ or its supplement.

Hence the locus of $C$ is the circle passing through the middle points of the sides of the triangle.

III. The intersections of pairs of tangents at $P$, $Q$, and $R$ lie on the lines joining the feet of the perpendiculars from the angular points on the sides of the triangle $PQR$.

Let $VCK$ (fig. 13) be the circle through the middle points of the sides of the triangle $PQR$, and let $VOK$ be a diameter of it.

Let $C$ be the centre of the hyperbola, and $D$ and $E$ feet of perpendiculars from the angular points $Q$, $P$ on the opposite sides.

Join $DE$ and $CK$; then since $VD$, $VE$ are equal chords and $VK$ a diameter, $DE$ is perpendicular to $VK$.

Let $DE$ intersect $VC$ in $T$, $VK$ in $N$.

From similar triangles $KCV$ and $TNV$,

$$CV.VT = KV.NV = VD^2.$$

But in the hyperbola, (fig. 12) if $T$ be the point of intersection of tangents at $P$ and $Q$, $A$ that of $CV$ and the curve, we have

$$CV.CT = CA^2.$$

Also in a rectangular hyperbola

$$PV^2 = CV^2 - CA^2$$

$$= CV^2 - CV.CT = CV.VT,$$

but $$PV = VD \text{ by (I)};$$

therefore the intersection of the tangents at $P$ and $Q$ lies on the straight line $DE$ joining the feet of the perpendiculars.

Similarly, the intersections of the tangents at $Q$ and $R$ and also at $P$ and $R$ lie on the lines joining the feet of the other perpendiculars. Hence the angular points of the triangle formed by the tangents at $P$, $Q$, and $R$ lie on the lines joining the feet of perpendiculars from the angular points on the sides of the triangle $PQR$.

IV. The perpendiculars drawn from the angular points of the triangle $PQR$ on the opposite sides intersect on the rectangular hyperbola.

Since the asymptotes of the hyperbola are at right angles to one another, therefore diameters drawn at right angles to one another are equal.

Therefore, if two chords of the hyperbola intersect at right angles, the rectangles contained by their segments are equal to one another.

Draw the perpendiculars $PE$, $QD$, and $RF$ (fig. 14) intersecting in $G$ and produce $PE$ to meet the circle described about $RPQ$ in $H$. Then since the angles $RGQ$ and $RPQ$ are equal to two right angles; therefore the angles $RGQ$, $RHQ$ are equal, and $EG=EH$; therefore the rectangle $GE.EP=$ the rectangle $RE.EQ$. Hence $G$ is a point on the rectangular hyperbola passing through $P$, $Q$, and $R$.

V. We have seen that the circle passing through the feet of the perpendiculars from the angular points on the sides of a triangle also passes through the middle points of those sides. Therefore the circle through $D$, $E$, and $F$ bisects $GP$, $GQ$, and $GR$ as well as the sides $PQ$, $QR$, and $RP$.

From (IV) and (V) alone it appears that since the six chords of the hyperbola are all bisected by the above circle, therefore it is the locus of the centre of the hyperbola.

The intersections of tangents at $P$ and $Q$, and at $R$ and $G$ lie on the straight line $DE$, *i.e.* the poles of $PQ$ and $RG$ with respect to the hyperbola lie on the straight line $DE$.

Therefore $F$ is the pole of $DE$ with respect to the rectangular hyperbola passing $P$, $Q$, $R$, and $G$; but $P$, $Q$, $R$, and $G$ are the centres of the escribed and inscribed circles of the triangle $DEF$.

Therefore if a rectangular hyperbola be so described that each angular point of a given triangle is the pole, with respect to it, of the opposite side, it will pass through the centres of the inscribed and escribed circles of the triangle; and its centre will be on the circumscribing circle.

# PROPERTY OF A CIRCLE TANGENTIAL TO THREE CIRCLES.

By *John Casey*, Scholar of Trinity College, Dublin.

*If a circle* W (fig. 15) *touch three circles* X, Y, Z, *the six points in which the radical axes of* X, Y, Z *intersect* W *may be divided into four sets of three points, each set of which are in involution with the three points of contact.*

For, let the points of contact be $S'$, $S''$, $S'''$, the radical axis of $X$, $Y$, intersect $W$ in $R'$, $R_1'$, the radical axis of $Y$, $Z$, in $R''$, $R_2''$, and the radical axis of $Z$, $X$, in $R'''$, $R_3'''$, let the radii of $W$, $X$, $Y$, $Z$ be $\rho$, $\rho'$, $\rho''$, $\rho'''$. Join $S'R'$, $S''R'$ intersecting $X$, $Y$, in $O'$, $O''$. Now since $S'$ is the centre of similitude of $W$ and $X$, we have

$$S'R' : R'O' :: \rho : \rho - \rho' ;$$

therefore $\quad S'R'^2 : S'R'.R'O' :: \rho : \rho - \rho',$

in like manner $\quad S''R'^2 : S''R'.R'O'' :: \rho : \rho - \rho'',$

and since $R'R_1'$ is the radical axis of $X$, $Y$, we have

$$S'R'.R'O' = S''R'.R'O'',$$

hence $\quad S'R'^2 : S''R'^2 :: \rho - \rho'' : \rho - \rho',$

in like manner $\quad S''R''^2 : S'''R''^2 :: \rho - \rho''' : \rho - \rho'',$

and $\quad S'''R'''^2 : S'R'''^2 :: \rho - \rho' : \rho - \rho''',$

hence $\quad S'R'.S''R''.S'''R''' = S''R'.S'''R''.S'R''' ;$

therefore the points $R'$, $R''$, $R'''$ are in involution with $S'$, $S''$, $S'''$. Exactly in the same way it may be proved that each of the sets of points

$$\left.\begin{matrix} R_1', \ R'', \ R''' \\ R_1', \ R_2'', \ R''' \\ R_1', \ R_2'', \ R_3''' \end{matrix}\right\} \text{ are in involution with } S', S'', S'''.$$

Q. E. D.

Model School, Kilkenny,
*Oct.* 29, 1861.

# ON A CERTAIN FORM OF EQUATIONS REPRESENTING A STRAIGHT LINE IN TRILINEAR COORDINATES.

It is not supposed that what is presented in the present paper will be new to all our readers; for many of them have, doubtless, already found the advantages of adopting in some cases equations of the form

$$\frac{\alpha-\alpha'}{\lambda}=\frac{\beta-\beta'}{\mu}=\frac{\gamma-\gamma'}{\nu}=\rho,$$

to represent a straight line in trilinear coordinates. But as the latest book which has treated of this method of coordinates, makes no reference whatever to these equations, and as no investigation of them appears ever to have been published, the following collection of the simple theorems on which any extensive application of them must be founded may save others the trouble of investigating them for themselves, and may induce them to give to the further pursuit of the subject those energies which might otherwise have been spent in re-discovering what is already known.

1. Suppose $l\alpha+m\beta+n\gamma=0$

to be the equation to any straight line referred to trilinear coordinates.

Then if $\alpha'$, $\beta'$, $\gamma'$ be the coordinates of any point in the line, we have

$$l\alpha'+m\beta'+n\gamma'=0.$$

Hence the line may be equally expressed by the equation

$$l(\alpha-\alpha')+m(\beta-\beta')+n(\gamma-\gamma')=0.$$

But we have

$$a(\alpha-\alpha')+b(\beta-\beta')+c(\gamma-\gamma')=0,$$

where $a$, $b$, $c$ are the lengths of the sides of the triangle of reference: and the last two equations are equivalent to

$$\frac{\alpha-\alpha'}{mc-nb}=\frac{\beta-\beta'}{na-lc}=\frac{\gamma-\gamma'}{lb-ma},$$

which will therefore also represent the line.

*Hence, the line* $l\alpha+m\beta+n\gamma=0$ *may be represented by the equations* $\frac{\alpha-\alpha'}{\lambda}=\frac{\beta-\beta'}{\mu}=\frac{\gamma-\gamma'}{\nu}$ *provided the constants* $\lambda$, $\mu$, $\nu$ *be chosen so as to satisfy the conditions*

$$\lambda:\mu:\nu=mc-nb:na-lc:lb-ma\;\ldots\ldots\ldots\text{(i)},$$

*and the constants* $\alpha'$, $\beta'$, $\gamma'$ *so as to satisfy the condition*

$$l\alpha'+m\beta'+n\gamma'=0.$$

2. We proceed to obtain the equations to a straight line in the form

$$\frac{\alpha-\alpha'}{\lambda}=\frac{\beta-\beta'}{\mu}=\frac{\gamma-\gamma'}{\nu},$$

without reference to the ordinary form.

Let $OP$ (fig. 16) be the straight line whose equations are required, $O$ a fixed point whose coordinates are $\alpha'$, $\beta'$, $\gamma'$, $ABC$ the triangle of reference, $\theta$, $\phi$, $\psi$ the angles which the line makes with the sides $BC$, $CA$, $AB$ respectively, $\alpha$, $\beta$, $\gamma$ the coordinates of any point $P$ in the line, and $\rho$ the distance $OP$.

Then if $OH$, $PM$ be perpendiculars upon $BC$, and $OF$ be drawn parallel to $BC$ to meet $PM$ in $F$, we have

$$\frac{PF}{OP}=\sin\theta,$$

that is, $$\frac{\alpha-\alpha'}{\rho}=\sin\theta,$$

so, $$\frac{\beta-\beta'}{\rho}=-\sin\phi=-\sin(C+\theta),$$

and, $$\frac{\gamma-\gamma'}{\rho}=\sin\psi=\sin(B-\theta).$$

Hence if $\lambda:\mu:\nu=\sin\theta:-\sin(C+\theta):\sin(B-\theta)$, then the equations

$$\frac{\alpha-\alpha'}{\lambda}=\frac{\beta-\beta'}{\mu}=\frac{\gamma-\gamma'}{\nu}\text{..............} (A),$$

will represent a straight line; and further, if $\lambda$, $\mu$, $\nu$ be not only proportional but absolutely equal to $\sin\theta$, $-\sin(C+\theta)$, $\sin(B-\theta)$, respectively, we may write

$$\frac{\alpha-\alpha'}{\lambda}=\frac{\beta-\beta'}{\mu}=\frac{\gamma-\gamma'}{\nu}=\rho,$$

where $\rho$ is the distance between the points $(\alpha\beta\gamma)$ and $(\alpha'\beta'\gamma')$.

3. The three equations of condition

$$\left.\begin{aligned}\lambda&=\sin\theta\\ \mu&=-\sin(C+\theta)\\ \nu&=\sin(B-\theta)\end{aligned}\right\}\text{....................} \text{(ii)}$$

will, on the elimination of $\theta$, lead to two equations among $\lambda$, $\mu$, $\nu$, and the angles of the triangle of reference.

Performing the elimination between the first and second and between the first and third, we obtain the two necessary and sufficient conditions

$$\lambda^2+\mu^2+2\lambda\mu\cos C=\sin^2 C,$$

$$\lambda^2+\nu^2+2\lambda\nu\cos B=\sin^2 B.$$

By symmetry, we must also have the equation

$$\mu^2+\nu^2+2\mu\nu\cos A=\sin^2 A,$$

but this does not express any further condition being obtainable from the two former by the elimination of $\lambda$.

Also, since

$$(\alpha-\alpha')\,a+(\beta-\beta')\,b+(\gamma-\gamma')\,c=0,$$

we get the further relation

$$\lambda a+\mu b+\nu c=0,$$

among $\lambda$, $\mu$, $\nu$, but this also must be implied in the former equations since they have been shewn to express the necessary and sufficient relations among $\lambda$, $\mu$, $\nu$.

Hence we arrive at the conclusion that the equations

$$\frac{\alpha-\alpha'}{\lambda}=\frac{\beta-\beta'}{\mu}=\frac{\gamma-\gamma'}{\nu}=\rho,$$

will represent a line passing through the point $(\alpha'\beta'\gamma')$, $\rho$ being the distance between this point and the point $(\alpha\beta\gamma)$, provided, and provided only, that $\lambda$, $\mu$, $\nu$ satisfy the conditions

$$\left.\begin{aligned} a\lambda+b\mu+c\nu &= 0\\ \mu^2+\nu^2+2\mu\nu\cos A &= \sin^2 A\\ \nu^2+\lambda^2+2\nu\lambda\cos B &= \sin^2 B\\ \lambda^2+\mu^2+2\lambda\mu\cos C &= \sin^2 C\end{aligned}\right\}\text{.............(iii)},$$

which are equivalent to only *two* independent equations.

4. The required conditions are given by any two of the four equations (iii), or by any two equations that can be formed by combining them. We proceed to obtain two such which are sometimes convenient.

Referring once more to the equivalent system (ii) we have

$$\mu^2=\sin^2\theta\cos^2 C+\cos^2\theta\sin^2 C+2\sin\theta\cos\theta\sin C\cos C$$

$$=\sin^2 C+\sin^2\theta\cos 2C+\sin\theta\cos\theta\sin 2C,$$

and similarly,

$$\nu^2=\sin^2 B+\sin^2\theta\cos 2B-\sin\theta\cos\theta\sin 2B;$$

therefore

$$\mu^2\sin 2B+\nu^2\sin 2C=2\sin B\sin C\sin(B+C)+\sin^2\theta\sin 2(B+C),$$

$$\text{or } \lambda^2\sin 2A+\mu^2\sin 2B+\nu^2\sin 2C=2\sin A\sin B\sin C\text{...(iv)}.$$

Again, $$\mu+\sin C\cos\theta+\sin\theta\cos C=0,$$

or $$\mu+\lambda\cos C=-\sin C\cos\theta,$$

so $$\nu+\lambda\cos B=\sin B\cos\theta;$$

therefore multiplying,

$$\mu\nu+\lambda\mu\cos B+\lambda\nu\cos C+\sin B\sin C(1-\lambda^2)+\lambda^2\cos B\cos C=0,$$

$$\text{or, } \mu\nu\sin A+\lambda\mu\sin A\cos B+\lambda\nu\sin A\cos C$$
$$+\sin A\sin B\sin C=\tfrac{1}{2}\lambda^2\sin 2A,$$

adding this to the two similar equations in $\nu$, $\lambda$ and $\lambda$, $\mu$, we obtain, in virtue of (iv),

$$2\mu\nu \sin A + 2\nu\lambda \sin B + 2\lambda\mu \sin C + 3 \sin A \sin B \sin C = \sin A \sin B \sin C,$$

or $\mu\nu \sin A + \nu\lambda \sin B + \lambda\mu \sin C + \sin A \sin B \sin C = 0$...(v).*

5. It will be observed that the condition

$$\lambda a + \mu b + \nu c = 0$$

is alone necessary in order that the equations ($A$) may represent a straight line, the ratios of $\lambda$, $\mu$, $\nu$, and not their absolute values being involved, but when we add the further equation

$$\ldots\ldots = \rho,$$

then in order that $\rho$ may be not merely proportional, but absolutely equal to the distance between the points we have to define the actual values of $\lambda$, $\mu$, $\nu$ by the second equation involved in (iii).

We have seen that when $\lambda$, $\mu$, $\nu$ are thus determined they are the sines of the angles which the straight line makes with the sides of the triangle of reference. We shall in future speak of them as the *direction-sines* of the straight line.

6. *To find the angle between the lines*

$$\frac{\alpha-\alpha'}{\lambda} = \frac{\beta-\beta'}{\mu} = \frac{\gamma-\gamma'}{\nu},$$

$$\frac{\alpha-\alpha'}{\lambda'} = \frac{\beta-\beta'}{\mu'} = \frac{\gamma-\gamma'}{\nu'}.$$

---

* The equations (iv) and (v) combined with the equations to the straight line, lead us at once to the expressions for the distance $\delta$ between any two points $(\alpha\beta\gamma)$, $(\alpha'\beta'\gamma')$, viz.:

$$\delta^2 = \frac{(\alpha-\alpha')^2 \sin 2A + (\beta-\beta')^2 \sin 2B + (\gamma-\gamma')^2 \sin 2C}{2 \sin A \sin B \sin C}$$

$$= -\frac{(\beta-\beta')(\gamma-\gamma') \sin A + (\gamma-\gamma')(\alpha-\alpha') \sin B + (\alpha-\alpha')(\beta-\beta') \sin C}{\sin A \sin B \sin C}.$$

(I.) If we suppose that $\lambda$, $\mu$, $\nu$, $\lambda'$, $\mu'$, $\nu'$ are equal (not merely proportional) to the direction-sines, we have

$$\sin\theta = \lambda, \qquad \sin\theta' = \lambda',$$
$$-\cos\theta = \frac{\mu + \lambda\cos C}{\sin C}, \qquad -\cos\theta' = \frac{\mu' + \lambda'\cos C}{\sin C},$$

$\theta$, $\theta'$ being the angles which the lines make with the line of reference $BC$. Let $D$ be the angle between the lines, then

$$\sin D = \sin(\theta' - \theta) = \frac{\lambda(\mu' + \lambda'\cos C) - \lambda'(\mu + \lambda\cos C)}{\sin C}$$
$$= \frac{\lambda\mu' - \lambda'\mu}{\sin C} = \frac{\mu\nu' - \mu'\nu}{\sin A} = \frac{\nu\lambda' - \nu'\lambda}{\sin B} \text{.........(vi)},$$

by symmetry. And likewise,

$$\cos D = \cos(\theta' - \theta) = \lambda\lambda' + \frac{(\mu + \lambda\cos C)(\mu' + \lambda'\cos C)}{\sin^2 C}$$
$$\left.\begin{aligned} &= \frac{\lambda\lambda' + \mu\mu' + (\lambda\mu' + \lambda'\mu)\cos C}{\sin^2 C} \\ &= \frac{\mu\mu' + \nu\nu' + (\mu\nu' + \mu'\nu)\cos A}{\sin^2 A} \\ &= \frac{\nu\nu' + \lambda\lambda' + (\nu\lambda' + \nu'\lambda)\cos B}{\sin^2 B} \end{aligned}\right\} \text{.....(vii)}.$$

(II.) If $\lambda$, $\mu$, $\nu$, $\lambda'$, $\mu'$, $\nu'$ be only proportional to the direction-sines of the straight lines these equations will no longer be true. But if we reduce the expressions in (vi) and (vii) so as to make them of zero dimensions, both in $\lambda$, $\mu$, $\nu$, and $\lambda'$, $\mu'$, $\nu'$, by means of the equations

$$1 = \frac{\sqrt{(\mu^2 + \nu^2 + 2\mu\nu\cos A)}}{\sin A} = \frac{\sqrt{(\nu^2 + \lambda^2 + 2\nu\lambda\cos B)}}{\sin B}$$
$$= \frac{\sqrt{(\lambda^2 + \mu^2 + 2\lambda\mu\cos C)}}{\sin C},$$

and the similar equations in $\lambda'$, $\mu'$, $\nu'$; then the expressions will not be altered if $\lambda$, $\mu$, $\nu$, or $\lambda'$, $\mu'$, $\nu'$ be changed in any constant ratio, and the equations thus formed will remain true in this case, when $\lambda$, $\mu$, $\nu$, $\lambda'$, $\mu'$, $\nu'$ are only proportional to the direction-sines.

Thus we shall get

$$\left.\begin{aligned}
\sin D &= \frac{(\mu\nu' - \mu'\nu)\sin A}{\sqrt{\{(\mu^2+\nu^2+2\mu\nu\cos A)(\mu'^2+\nu'^2+2\mu'\nu'\cos A)\}}} = \&c.\\
\cos D &= \frac{\mu\mu'+\nu\nu'+(\mu\nu'+\mu'\nu)\cos A}{\sqrt{\{(\mu^2+\nu^2+2\mu\nu\cos A)(\mu'^2+\nu'^2+2\mu'\nu'\cos A)\}}} = \&c.\\
&\text{whence}\\
\cot D &= \frac{\mu\mu'+\nu\nu'+(\mu\nu'+\mu'\nu)\cos A}{\mu\nu'-\mu'\nu} = \&c.
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\text{(viii)}.$$

7. *To deduce expressions for the angle between the lines whose equations are given in the form*

$$l\alpha + m\beta + n\gamma = 0,$$
$$l'\alpha + m'\beta + n'\gamma = 0.$$

Referring to equations (i) we see that these two lines may be expressed by the equations

$$\frac{\alpha-\alpha'}{\lambda} = \frac{\beta-\beta'}{\mu} = \frac{\gamma-\gamma'}{\nu},$$
$$\frac{\alpha-\alpha'}{\lambda'} = \frac{\beta-\beta'}{\mu'} = \frac{\gamma-\gamma'}{\nu},$$

where $(\alpha'\beta'\gamma')$ is their point of intersection, and $\lambda$, $\mu$, $\nu$, $\lambda'$, $\mu'$, $\nu'$ are given by the equations

$$\begin{aligned}
\lambda &= m\sin C - n\sin B, & \lambda' &= m'\sin C - n'\sin B,\\
\mu &= n\sin A - l\sin C, & \mu' &= n'\sin A - l'\sin C,\\
\nu &= l\sin B - m\sin A, & \nu' &= l'\sin B - m'\sin A,
\end{aligned}$$

hence substituting

$$\mu\nu' - \mu'\nu = \frac{\sin^2 A}{a}\{a(mn'-m'n) + b(nl'-n'l) + c(lm'-l'm)\},$$

$$\begin{aligned}
&\mu^2+\nu^2+2\mu\nu\cos A\\
&= \sin^2 A\,[l^2+m^2+n^2-2ln\cos A-2nl\cos B-2lm\cos C],
\end{aligned}$$

$$\begin{aligned}
&\mu\mu'+\nu\nu'+(\mu\nu'+\mu'\nu)\cos A\\
&= \frac{\sin^2 A}{a^2}\,[(na-lc)(n'a-l'c)+(lb-ma)(l'b-m'a)\\
&\qquad\qquad + \cos A\,\{(na-lc)(l'b-m'a)+(lb-ma)(n'a-l'c)\}]\\
&= \sin^2 A\,\{ll'+mm'+nn'-(l'n+ln')\cos B\\
&\qquad\qquad -(lm'+l'm)\cos C-(mn'+m'n)\cos A\}.
\end{aligned}$$

The function $l^2+m^2+n^2-2mn\cos A-2nl\cos B-2lm\cos C$ where $l$, $m$, $n$ are any constants, and $A$, $B$, $C$ are the angles of the triangle of reference, is of such constant occurrence that it is convenient to represent it by the form $\{l, m, n\}^2$.

Thus we obtain

$$\left.\begin{aligned}
&\sin D=\frac{\sin A\,(mn'-m'n)+\sin B\,(nl'-n'l)+\sin C\,(lm'-l'm)}{\{l, m, n\}\ \{l', m', n'\}}\\
&\cos D\\
&=\frac{ll'+mm'+nn'-(mn'+m'n)\cos A-(nl'+n'l)\cos B-(lm'+l'm)\cos C}{\{l, m, n\}\ \{l', m', n'\}}\\
&\text{and } \cot D\\
&=\frac{ll'+mm'+nn'-(mn'+m'n)\cos A-(nl'+n'l)\cos B-(lm'+l'm)\cos C}{(mn'-m'n)\sin A+(nl'-n'l)\sin B+(lm'-l'm)\sin C}
\end{aligned}\right\}$$

..............(ix).

COR. 1. The two straight lines

$$l\alpha+m\beta+n\gamma=0,$$
$$l'\alpha+m'\beta+n'\gamma=0,$$

are at right angles to one another, provided

$$ll'+mm'+nn'-(mn'+m'n)\cos A$$
$$-(nl'+n'l)\cos B-(lm'+l'm)\cos C=0.$$

COR. 2. If the equation

$$u\alpha^2+v\beta^2+w\gamma^2+2u'\beta\gamma+2v'\gamma\alpha+2w'\alpha\beta=0$$

represent two straight lines, they are at right angles to one another, provided

$$u+v+w-2u'\cos A-2v'\cos B-2w'\cos C=0.$$

OBS. It is convenient to term the ratios $\frac{l}{\{l, m, n\}}$, $\frac{m}{\{l, m, n\}}$, $\frac{n}{\{l, m, n\}}$ the direction-ratios of the straight line whose equation is $l\alpha+m\beta+n\gamma=0$.

In other words, if the constants in this equation be made to satisfy the relation $\{l, m, n\}=1$, they are called the direction ratios of the straight line.

8. The direction-sines of any straight line in terms of its direction-ratios will be [compare (i)]

$$\left.\begin{aligned}\lambda &= m\sin C - n\sin B\\ \mu &= n\sin A - l\sin C\\ \nu &= l\sin B - m\sin A\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(\text{x}),$$

for we have seen in Art. 7, that these substitutions give

$$\frac{\mu^2+\nu^2+2\mu\nu\cos A}{\sin^2 A}=\{l,\ m,\ n\}^2.$$

But by hypothesis $l$, $m$, $n$ are direction-ratios, and therefore

$$\{l,\ m,\ n\}=1;$$

therefore $\mu^2+\nu^2+2\mu\nu\cos A=\sin^2 A$,

(and similar equations), and therefore $\lambda$, $\mu$, $\nu$ are direction-sines, by Art. 5.

*Conversely*, the direction-ratios in terms of the direction-sines are given by

$$\left.\begin{aligned}l &= \frac{a}{\sin A}(\nu\beta'-\mu\gamma')\div 2\Delta\\ m &= \frac{b}{\sin B}(\lambda\gamma'-\nu\alpha')\div 2\Delta\\ n &= \frac{c}{\sin C}(\mu\alpha'-\lambda\beta')\div 2\Delta\end{aligned}\right\}\ldots\ldots\ldots\ldots(\text{xi}),$$

where $\Delta$ = area of the triangle.

For $l(\alpha-\alpha')+m(\beta-\beta')+n(\gamma-\gamma')=0$;

therefore $l\lambda+m\mu+n\nu=0$,

also $l\alpha'+m\beta'+n\gamma'=0$;

therefore $\dfrac{l}{\nu\beta'-\mu\gamma'}=\dfrac{m}{\lambda\gamma'-\nu\alpha'}=\dfrac{n}{\mu\alpha'-\lambda\beta'}=\dfrac{1}{\kappa}$ suppose.

Substituting in the first of equations (x)

$$\begin{aligned}\kappa\lambda &= (\lambda\gamma'-\nu\alpha')\sin C-(\mu\alpha'-\lambda\beta')\sin B\\ &= \lambda(\gamma'\sin C+\beta'\sin B)-\alpha'(\mu\sin B+\nu\sin C)\\ &= 2\Delta\lambda\frac{\sin A}{a}-\alpha'(\lambda\sin A+\mu\sin B+\nu\sin C);\end{aligned}$$

therefore $\kappa=2\Delta.\dfrac{\sin A}{a}$, whence the above values of $l$, $m$, $n$ can be immediately written down.

9. The expression in equations (ix) for $\sin D$ may be written in the form

$$\frac{l(m'\sin C-n'\sin B)+m(n'\sin A-l'\sin C)+n(l'\sin B-n'\sin A)}{\{l,\ m,\ n\}\ \{l',\ m',\ n'\}}.$$

Suppose $l'$, $m'$, $n'$ the direction-ratios of the equation in the ordinary form to the line whose direction-sines are $\lambda$, $\mu$, $\nu$, then we have, by (x),

$$\sin D=\frac{l\lambda+m\mu+n\nu}{\{l,\ m,\ n\}} \text{ ............... (xii)},$$

an expression for the angle between two lines, one of which is expressed in the form

$$l\alpha+m\beta+n\gamma=0,$$

and the other in the form

$$\frac{\alpha-\alpha'}{\lambda}=\frac{\beta-\beta'}{\mu}=\frac{\gamma-\gamma'}{\nu}=\rho.$$

We proceed to a few examples of the application of the equations which we have investigated.

10. *To find the perpendicular distance of the point* ($fgh$) *from the straight line whose equation is*

$$l\alpha+m\beta+n\gamma=0 \text{ .................... (1)}.$$

Let
$$\frac{f-\alpha}{\lambda}=\frac{g-\beta}{\mu}=\frac{h-\gamma}{\nu}=\rho \text{ ............... (2)}$$

be the equation to the perpendicular from ($fgh$) upon the given line. Then the length of the intercept is given by

$$l(f-\lambda\rho)+m(g-\mu\rho)+n(h-\nu\rho)=0,$$

or
$$\rho=\frac{lf+mg+nh}{l\lambda+m\mu+n\nu}.$$

But since the lines (1) and (2) are at right angles to one another, we have, by (xii),

$$l\lambda+m\mu+n\nu=\{l,\ m,\ n\};$$

therefore the perpendicular distance is

$$\frac{lf+mg+nh}{\{l, m, n\}}.$$

11. *To find the equations to the perpendicular from* $(fgh)$ *on the straight line* $l\alpha+m\beta+n\gamma=0$.

Let $$\frac{\alpha-f}{\lambda}=\frac{\beta-g}{\mu}=\frac{\gamma-h}{\nu}$$

be the equations to the perpendicular required, then by the condition of perpendicularity (Art. 9),

$$l\lambda+m\mu+n\nu=\{l, m, n\},$$

or, making the equation homogeneous in $\lambda$, $\mu$, $\nu$,

$$(l\lambda+m\mu+n\nu)\sin C=\{l, m, n\}\sqrt{(\lambda^2+\mu^2+2\lambda\mu\cos C)},$$

or $\lambda(l\sin C-n\sin A)+\mu(m\sin C-n\sin B)$

$$=\{l, m, n\}\sqrt{(\lambda^2+\mu^2+2\lambda\mu\cos C)},$$

squaring and transposing, the coefficient of $\lambda^2$ is

$$\{l, m, n\}^2-l^2\sin^2 C-n^2\sin^2 A+2ln\sin A\sin C$$
$$=(m-n\cos A-l\cos C)^2.$$

Similarly, the coefficient of $\mu^2$ is $=(l-m\cos C-n\cos B)^2$,

and, since we know there cannot be two different solutions, the coefficient of $\lambda\mu$ *without calculation* must be

$$-2(l-m\cos C-n\cos B)(m-n\cos A-l\cos C).$$

Hence

$$\frac{\lambda}{l-m\cos C-n\cos B}=\frac{\mu}{m-n\cos A-l\cos C}=\frac{\nu}{n-l\cos B-m\cos A},$$

which determine $\lambda$, $\mu$, $\nu$.

From this it follows that the equations to the straight line drawn through the point $(fgh)$ perpendicular to the straight line $l\alpha+m\beta+n\gamma=0$ are

$$\frac{\alpha-f}{l-m\cos C-n\cos B}=\frac{\beta-g}{m-n\cos A-l\cos C}=\frac{\gamma-h}{n-l\cos B-m\cos A}.$$

COR. The *tangent* at the point $(\alpha'\beta'\gamma')$ to the conic

$$f(\alpha\beta\gamma)=0$$

is given by

$$\alpha \frac{df}{d\alpha'} + \beta \frac{df}{d\beta'} + \gamma \frac{df}{d\gamma'} = 0.$$

Hence the *normal* at $(\alpha'\beta'\gamma')$ will be expressed by the equations

$$\frac{\alpha - \alpha'}{\frac{df}{d\alpha'} - \frac{df}{d\beta'}\cos C - \frac{df}{d\gamma'}\cos B} = \frac{\beta - \beta'}{\frac{df}{d\beta'} - \frac{df}{d\gamma'}\cos A - \frac{df}{d\alpha'}\cos C}$$
$$= \frac{\gamma - \gamma'}{\frac{df}{d\gamma'} - \frac{df}{d\alpha'}\cos B - \frac{df}{d\beta'}\cos A}.$$

We forbear to trespass further on these pages at present: we reserve till the Second Number of the *Messenger* the applications of the formulæ we have investigated to determine the nature of the conic represented by the general equation of the second degree,—to find the equations to its foci and axes,—and to consider the relations connecting its conjugate diameters.

W.

(*To be continued.*)

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# GEOMETRICAL PROPERTIES OF THE MOTION OF A PARTICLE UNDER THE ACTION OF GRAVITY DESCRIBING A VERTICAL CIRCLE.

By *John Casey*, Scholar of Trinity College, Dublin.

LEMMA. *Let* X, Y (fig. 17) *be two circles;* AB, A'B' *two positions of a variable chord of* Y, *which is a tangent to* X*;* p, p', P, P' *perpendiculars from* A, A', B, B' *on the radical axis of* X *and* Y*; then the chord* AA' : *chord* BB' :: $p^{\frac{1}{2}} + p'^{\frac{1}{2}} : P^{\frac{1}{2}} + P'^{\frac{1}{2}}$.

For, let $C$, $C'$ be the points of contact of $AB$, $A'B'$ with $X$, and $O$ the point of intersection of $AB$, $A'B'$. Now the triangles $AA'O$, $BB'O$ are similar; therefore

$$AA' : BB' :: AO + A'O : BO + B'O$$
$$:: AC + A'C' : BC + B'C' :: p^{\frac{1}{2}} + p'^{\frac{1}{2}} : P^{\frac{1}{2}} + P'^{\frac{1}{2}}$$

(see Chasles' *Géométrie Supérieure*, Art. 728). Q.E.D.

Let $X$, $Y$ (fig. 18) be two vertical circles in the same plane; $RA$ their radical axis; $ab$, $a'b'$ two positions indefinitely near of a chord of $Y$ which is a tangent to $X$; then by the lemma, $aa' : bb'$ :: the square root of the perpendicular from $a$ on $RA$ : square root of the perpendicular from $b$ on $RA$. Hence, if a material particle $m$ be projected from $a$ with the velocity due to the height from $RA$, it is evident that the indefinitely small interval of time in which it describes the elementary arc $aa'$, is equal to the indefinitely small interval it will take to describe the elementary arc $bb'$: to these equal instants add the time of describing the arc $a'b$, and we have the time of describing the arc $ab$ equal to the time of describing the arc $a'b'$. Hence we have the following theorem:

*If any number of tangents be drawn to* X, *the times of the particle* m *describing the arcs they intercept on* Y *are all equal.*

Cor. 1. *If* Z *be another circle coaxial with* X *and* Y, *and chords of* Y *be tangents to* Z, *the times of describing the arcs they intercept on* Y *will be all equal.*

Cor. 2. *When the circles intersect we have a property of the motion of a simple pendulum. When they touch at the highest point, we have the case of a particle projected from any point of a vertical circle with the velocity due to a fall from the highest point, and when the circles do not intersect we have the case of a particle making complete revolutions in a vertical circle.*

Cor. 3. *Hence we see the dynamical interpretation of the theorem given by Poncelet, Propriétés Projectives*, p. 323.

In a future communication I shall shew that the analytical statement is no less important.

Model School, Kilkenny,
*October* 29, 1861.

# ON THE EQUATIONS OF MOTION OF A PLANET REFERRED TO MOVING AXES.

By *C. H. H. Cheyne, B.A.*, Scholar of St. John's College, Cambridge.

THE object of this paper is to shew that the equations of motion of a planet, referred to its radius vector and a perpendicular to it in the plane of its orbit, take the same forms as if this plane were at rest.*

Let $X$, $Y$, $Z$ be the accelerations, $u$, $v$, $w$ the velocities of a planet, parallel to three rectangular axes originating in the Sun's centre, and moving with angular velocities $\omega_1$, $\omega_2$, $\omega_3$ about their instantaneous positions. Then we have

$$X = \frac{du}{dt} - v\omega_3 + w\omega_2,$$

$$Y = \frac{dv}{dt} - w\omega_1 + u\omega_3,$$

$$Z = \frac{dw}{dt} - u\omega_2 + v\omega_1,$$

where

$$u = \frac{dx}{dt} - y\omega_3 + z\omega_2,$$

$$v = \frac{dy}{dt} - z\omega_1 + x\omega_3,$$

$$w = \frac{dz}{dt} - x\omega_2 + y\omega_1.$$

In these expressions $\omega_1$, $\omega_2$, $\omega_3$ may be determined at pleasure; we propose so to determine them that the plane of $xy$ may coincide with the plane of the orbit.

Now the plane of the orbit being defined to be the plane passing through the radius vector and direction of motion of the planet, in order that the plane of $xy$ may coincide with it, we must have $z=0$, $w=0$; also since $z=0$ always, it follows that $\frac{dz}{dt}=0$. The expressions for $X$ and $Y$ now become

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* See a Memoir by Professor Donkin on "The Differential Equations of Dynamics," *Philosophical Transactions*, 1855.

$$X=\frac{du}{dt}-v\omega_3,$$

$$Y=\frac{dv}{dt}+u\omega_3,$$

where $$u=\frac{dx}{dt}-y\omega_3,$$

$$v=\frac{dy}{dt}+x\omega_3.$$

Now these are the same in form as would be obtained by putting $\omega_1=0$, $\omega_2=0$, *i.e.* as if the plane of $xy$ were at rest.

Since $\omega_3$ is undetermined, let it $=0$, in which case the axes of $x$ and $y$ are said to be *fixed in the plane of the orbit.* We then have

$$X=\frac{du}{dt}=\frac{d^2x}{dt^2}, \quad Y=\frac{dv}{dt}=\frac{d^2y}{dt^2},$$

as if the axes were absolutely at rest. The condition $\omega_3=0$ may be looked upon as a definition of the term "fixed in the orbit."

If in these expressions we put $x=r\cos\theta$, $y=r\sin\theta$, we find for the accelerations parallel and perpendicular to the radius vector of the planet

$$\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2, \text{ and } \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right),$$

where $\theta$, being the inclination of the radius vector to the axis of $x$, is measured from a line fixed in the plane of the orbit. Hence the equations of motion of a planet in these directions are the same in form as if the motion were in one plane.

It may be interesting to shew that lines fixed in the plane of the orbit revolve about the radius vector of the planet as an instantaneous axis. For, since $\omega_3 = 0$, the equations to this axis are

$$\frac{\xi}{\omega_1} = \frac{\eta}{\omega_2}, \quad \zeta = 0,$$

but since $w = 0$ and $\frac{ds}{dt} = 0$,

$$\frac{\omega_1}{x} = \frac{\omega_2}{y};$$

hence the equations become

$$\frac{\xi}{x} = \frac{\eta}{y}, \quad \zeta = 0,$$

whence the instantaneous axis coincides with the radius vector of the planet.

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## NOTE ON CURVATURE.

Geometrical proof of the proposition that principal sections are sections along which consecutive normals intersect.

Let *ON* (fig. 19) be the normal, *OA* a principal section (*i.e.*, a normal section of maximum or minimum curvature), *OP* the tangent line to *OA*.

Draw *LPM* in the tangent plane at *O* perpendicular to *OP*. Draw *PA*, *LB*, *MC* perpendicular to the tangent plane to meet the surface (*L*, *M* being points very near to *P*).

Then *OA* is a section of maximum or minimum curvature;

therefore $PA >$ or $<$ both $LB$, $MC$;

therefore *B*, *A*, *C* are situated in a curve of which the ordinate *PA* is a maximum or minimum, therefore *AT* the tangent at *A* is parallel to *LM*, and therefore perpendicular to the plane *POA*, but the normal at *A* is perpendicular to *AT*, and therefore is in the plane *POA*, and therefore will intersect *ON*.

W. H.

# HINTS ON THE THEORY OF GROUPS.

By the *Rev. T. P. Kirkman, M.A., F.R.S.*, Honorary Member of the Literary and Philosophical Societies of Manchester and Liverpool.

1. The object of this and one or two following papers is to introduce the learner to the theory of groups, without however undertaking to write a regular elementary treatise on the subject.

I shall endeavour here to give the student a clear idea, 1st of what a group is, and 2nd of the use to which a group is applied.

A serviceable way of doing this will be, to write down the groups that can be made with two, three, and four elements; next to demonstrate a few preliminary theorems; and then to indicate how *m*-valued functions may be constructed on the groups.

2. The only group possible with two elements is

$$\begin{array}{l}12\\21 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (A),\end{array}$$

which comprises all the 1.2 permutations of the elements.

With three elements we can make the groups

$$\begin{array}{lll}123 & 123 & 123\\213, & 321, & 132 \ldots\ldots\ldots\ldots (a_1a_2a_3),\end{array}$$

$$\begin{array}{l}123\\231\\312 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (B),\end{array}$$

and, finally,

$$\begin{array}{ll}123 & 132\\231 & 321\\312, & 213 \ldots\ldots\ldots\ldots\ldots\ldots (C),\end{array}$$

of which the last comprises all the 1.2.3 permutations of the elements.

With four elements we can make the groups

$$\begin{array}{llllll}1234 & 1234 & 1234 & 1234 & 1234 & 1234\\2134, & 1243, & 3214, & 1324, & 1432, & 4231 \ldots (e_1e_2 \ldots e_6),\end{array}$$

$$\begin{array}{llll} 1234 & 1234 & 1234 & \\ 2143, & 3412, & 4321 & \ldots\ldots\ldots(DD'D''), \end{array}$$

$$\begin{array}{lllll} 1234 & 1234 & 1234 & 1234 & \\ 2314 & 2431 & 3241 & 1342 & \\ 3124, & 4132, & 4213, & 1423 & \ldots(f_1f_2f_3f_4), \end{array}$$

$$\begin{array}{lllll} 1234 & 1234 & 1234 & 1234 & \\ 2314 & 2431 & 3241 & 1342 & \\ 3124, & 4132, & 4213, & 1423, & \\[1ex] 1324 & 1432 & 1243 & 1243 & \\ 3214 & 4231 & 4231 & 1432 & \\ 2134, & 2134, & 3214, & 1324 & \ldots(g_1g_2g_3g_4), \end{array}$$

$$\begin{array}{ll} 1234 & \\ 2143 & \\ 3412 & \\ 4321 & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(E), \end{array}$$

$$\begin{array}{llll} 1234 & 1234 & 1234 & \\ 2143 & 4321 & 3412 & \\ 1243 & 1324 & 1432 & \\ 2134, & 4231, & 3214 & \ldots\ldots\ldots(FF'F''), \end{array}$$

$$\begin{array}{llll} 1234 & 1234 & 1234 & \\ 2341 & 3142 & 4312 & \\ 3412 & 4321 & 2143 & \\ 4123, & 2413, & 3421 & \ldots\ldots(G'G''G'''), \end{array}$$

$$\begin{array}{llll} 1234 & 1234 & 1234 & \\ 2341 & 3142 & 4312 & \\ 3412 & 4321 & 2143 & \\ 4123, & 2413, & 3421, & \\[1ex] 1432 & 1324 & 1243 & \\ 4321 & 2143 & 3412 & \\ 3214 & 4231 & 2134 & \\ 2143, & 3412, & 4321 & \ldots\ldots\ (HH'H''), \end{array}$$

$$\begin{array}{llll} 1234 & 2314 & 3124 & \\ 2143 & 1423 & 4213 & \\ 3412 & 4132 & 1342 & \\ 4321, & 3241, & 2431 & \ldots\ldots\ldots\ldots\ldots(I), \end{array}$$

$$\begin{array}{lllllll} 1234 & 2314 & 3124 & 1324 & 3214 & 2134 & \\ 2143 & 1423 & 4213 & 2413 & 4123 & 1243 & \\ 3412 & 4132 & 1342 & 3142 & 1432 & 4312 & \\ 4321, & 3241, & 2431, & 4231, & 2341, & 3421 & \ldots(J), \end{array}$$

of which the last comprises all the 1.2.3.4 permutations of the elements.

The student may take my word, at present, that the above are all the groups which have among their arrangements the natural order 123... of the elements, that can be made with four or with fewer letters.

3. *Substitutions.* Any permutation $A$ of the $N$ elements 123...$N$, being written under any permutation $B$ of the same elements, we have the *substitution* $\frac{B}{A}$, which always has reference to a subject $C$, on which we operate with the substitution. This $C$ may be any permutation of the same $N$ elements, different or not from $A$ or $B$. The effect of $\frac{B}{A}$ upon $C$ is to put in $C$, for any element $m$ in it, that which stands over $m$ in $\frac{B}{A}$. We write the subject of operation on the right of the operating substitution, and we thus express that $D$ is the result:

$$\frac{B}{A}\,C=D.$$

Thus we have

$$\frac{2143}{4123}\,2341=4321,$$

$$\frac{2143}{4321}\,2341=4123.$$

We see that

$$\frac{2143}{4321}=\frac{2431}{4213}=\frac{2314}{4132}=\frac{3412}{1234},\text{ etc.}$$

are the same substitution; for the same element stands over $m$ in all of them.

Hence we can write every possible substitution with 1234..., the natural order, below. It is usual to call this order *unity*.

We can thus consider any *permutation* $A$ of the $N$ elements as a *substitution* whose lower term is unity, which however needs not to be written under $A$; but is always understood, if we speak of $A$ as a *substitution*, not merely a *permutation*. That is

$$A = \frac{A}{123\ldots} = \frac{A}{1}$$

all represent the same substitution.

The groups above written are all to be considered as groups of substitutions; that is, every arrangement in them is conceived to be written over *unity*.

The subject ($C$) on which we operate with the substitution $A$ may be any number $C_1 C_2 C_3 \ldots$ of arrangements of the $N$ elements. The result of operation is always an equal number of arrangements. Thus,

$$A.(C_1 + C_2 + C_3 \ldots) = D_1 + D_2 + D_3 \ldots,$$

where $D_1$ differs from $C_1$ only in the elements disturbed by the substitution $A$.

The subject of operation may be any algebraic function of the $N$ variables

$$x_1 x_2 x_3 \ldots x_N.$$

*e.g.* This function may be ($N = 4$)

$$x_1^3 x_2 + x_3^2 + x_4 = x_1^3 x_2 x_3^0 x_4^0 + x_1^0 x_2^0 x_3^2 x_4^0 + x_1^0 x_2^0 x_3^0 x_4.$$

Operation on this subject with $A = 1234$ makes no change. Operation on it with $A' = 2134$ gives the result

$$x_2^3 x_1 + x_3^2 + x_4;$$

for $A'$ disturbs neither $x_3$ nor $x_4$. Operation on the same subject with $A'' = 4123$ gives

$$x_4^3 x_1 + x_2^2 + x_3.$$

4. *Product of two substitutions.* The subject being $C$, and $A'C$ the result of the operation with $A'$, we can operate on this result with $A''$, obtaining the result

$$A''.A'C = A''A'.C.$$

The student can very easily satisfy himself that these are identical results: or that it comes to the same thing, whether we first put $n$ for $m$ in $C$ by the operation $A'$, and then for $n$ in $A'C$ put $k$ by the operation $A''$, or whether we first put $k$ for $n$ in $A'$ by operating on $A'$ with $A''$, and then in $C$ put $k$ for $m$ by the operation $A''A'$ upon $C$.

By this process of operation with $A''$ on the result $A'C$, we have operated on $C$ by what is called *the product* $A''A'$ *of the substitutions* $A''$ *and* $A'$.

This product $A''A'$ is the result of operating on a substitution with a substitution. We can write and examine this result apart from the consideration of the subject $C$; or, what is the same thing, we can take a substitution for the subject $C$.

It is important to be able to write down the product $F$ of the substitutions $A$ and $B$, both referred to *unity* as the lower term, the number of elements being $N$, viz.

$$AB = F.$$

a. The simple rule is—*Pronounce the left-hand substitution* $A$, *and write down its consecutive elements exactly in the order in which you see* 1234...$N$ *disposed in* $B$.

Thus we have (*vide* group $G''$)

$$3142.1234 = \quad 3142 \quad = 1234.3142,$$
$$3142.3142 = 4321 = (3142)^2,$$
$$3142.4321 = 2413 = (3142)^3 = 4321.3142,$$
$$3142.2413 = 1234 = (3142)^4 = 2413.3142.$$

Again, we have (*vide* group $H'$)

$$3142.1324 = 3412 = 4321.2143,$$
$$1324.3142 = 2143 = 4321.3412.$$

By this example we see that $AB$ and $BA$ are not always the same product.

b. We may, if we choose, consider the product $AB = F$ to be the result of operation with $B$ on the right hand upon $A$ on the left.

The same result $F$ is obtained by this rule: *Pronounce the successive elements abcd... of the right-hand factor* $B$, *and* *write down in order from left to right the* $a^{\text{th}}$, *the* $b^{\text{th}}$, *the* $c^{\text{th}}$... *elements of* $A$.

It is easy to verify these rules on the examples above given.

5. *Groups of substitutions.* We can now understand the definition of a group. It is simply this:

*Any number* $K$ *of substitutions, such that the product* ($AB$, *or* $BA$, *or* $AA$) *of any two of them is among them, forms a group of substitutions.*

This condition is fulfilled in all the groups written in the second article.

If the group contains the arrangement *unity*, it is called a *model group;* and *every group* can be written to contain *unity*. But there is no necessity that *unity* should be found in it. For example (*vide* group $F$)

$$\begin{array}{lcl} 1234 & & 2431 \\ 2143 & & 1342 \\ 1243 & & 2341 \\ 2134 & \text{and} & 1432 \end{array}$$

are the same group of substitutions, though not the same system of permutations. We have

$$\frac{1342}{2431} = \frac{2143}{1}, \quad \frac{2341}{2431} = \frac{1243}{1},$$

$$\frac{2341}{2431} = \frac{1243}{1}, \quad \frac{1432}{2431} = \frac{2134}{1}.$$

It follows from the definition of a model group, that every power of every substitution in it is one of the group. We must have, if the substitution $A$ be in the group,

$$AA = B = A^2,$$

$$AB = C = A^3,$$

$$AC = D = A^4,$$

etc.,

when $BCD$... are all substitutions of the group.

Hence there will be of necessity a power $(k+1)$ of $A$ such that

$$A^{k+1}=A,$$

or

$$A^{k}\ =1.$$

*This substitution $A$ is said to be of the $k^{\text{th}}$ order.*

*A group of $K$ substitutions is said to be a group of the $K^{\text{th}}$ order.*

It follows also that every negative power of a substitution is one of the group. For

$$A^{-m}=A^{k-m}=\frac{1}{A^{m}}=J$$

is a substitution of the group. But it does not follow that every root of a substitution is in the group. If

$$MM=A,$$

$M$ may or may not be in a given group which contains $A$. But no group contains $M$ which does not also contain $A$.

6. *Derangements of a group.*

Let $G$ be any model group of the $K^{\text{th}}$ order, made with $N$ elements, which does not contain the substitution $M$. The group being

$$G=1+A_1+A_2+\ldots+A_{K-1},$$

we can write the product

$$GM=M+A_1M+A_2M+\ldots+A_{K-1}M.$$

If $G$ and $GM$ be each written in one column, we find that they differ only in the horizontal order of entire vertical rows; for, by the rule (4, b), we see that the $m^{\text{th}}$ element of every substitution of $G$ is read in one and the same vertical row of $GM$.

*We call $GM$ the derangement of $G$ by $M$.*

$G$ and $GM$, as may be seen by the example of the preceding article, are the same group of substitutions, if the substitutions are written with two terms in the form $\frac{A}{B}$, taken both from $G$ or both from $GM$. But if all the arrangements of both are written over unity we can prove that $G$ and $GM$ have no substitution in common.

THEOREM I. *The group $G$ has no substitution in common with $GM$, its derangement by $M$.*

For let it be supposed that $A_pM = A_q$, and let

$$A_p^r = 1 = A_p^{r-1}A_p;$$

it follows that $M = A_p^{r-1}A_q = A_s$, or $M$ is a substitution of $G$, contrary to hypothesis.

THEOREM II. *The derangement of $G$ by $M$ is the derangement of $G$ by $A_mM$, $A_m$ being any substitution of $G$.* For

$$GA_mM = A_mM + A_1A_mM + A_2A_mM + \dots + A_{K-1}A_mM.$$

which, by the definition of a group, (5), is merely $GM$ written in a different order.

THEOREM III. *The entire group of $\underline{|N}$ substitutions can be written as the model group $G$ followed by $\frac{\underline{|N}}{K} - 1$ derangements of $G$.*

For let $M_1$ be any substitution not found either in $G$ or in $GM$, unity being everywhere the lower term. We have

$$GM_1 = M_1 + A_1M_1 + A_2M_1 + \dots + A_{K-1}M_1,$$

of which no substitution is found either in $G$ or in $GM$. For if $A_cM = A_dM_1$, let $A_d^t = 1$; then it follows that

$$A_d^{t-1}A_cM = AM = M_1,$$

*i.e.* $M_1$ is a substitution of $GM$, contrary to hypothesis.

Whereupon $G$, $GM$, $GM_1$ have no substitution in common.

Let $M_2$ be a substitution not found in $G$, $GM$, or $GM_1$. We prove exactly in the same way that both $A_cM = A_dM_2$ and $A_cM_1 = A_dM_2$ are impossible, and that $G$, $GM$, $GM_1$, $GM_2$, have no substitution in common.

In this way, by adding to those already formed the derangement $GM_3$ of $G$ by $M_3$, which is not found in the preceding, we shall finally exhaust all the $\underline{|N}$ arrangements of $N$ elements, and thus prove the theorem.

7. *Derivates of a Group.* The product

$$MG_1 = M + MA_1 + MA_2 + \ldots + MA_{K-1},$$

of $G$, and any substitution $M$ not in $G$, is called the *derivate of G by M.*

THEOREM IV. *No substitution of $MG$, the derivate of $G$ by $M$, is a substitution of $G$.*

For if $MA_c = A_d$, let $A_c^r = 1$, then $M = A_d A_c^{r-1} = A_e$, is a substitution of $G$, contrary to hypothesis.

THEOREM V. *The derivate of the model group $G$ by $M$ is the derivate of $G$ by $MA_c$, $A_c$ being any substitution of $G$.*

For $\quad MA_cG = MA_c\, MA_cA_1 + MA_cA_2 + \ldots + MA_cA_{K-1},$

which, by the definition of a group, is merely $MG$ in a different order.

We learn thus that the derangement $G'$ of $G$ is the derangement of $G$ by any substitution of $G'$, and that the derivate $G_1$ of $G$ is the derivate of $G$ by any substitution of $G_1$, unity being taken always for the lower term of the substitution.

THEOREM VI. *The entire group of $\underline{|N}$ arrangements of $N$ elements can be written as the model group $G$ of the $K^{\text{th}}$ order, followed by $\frac{\underline{|N}}{K} - 1$ derivates of $G$.*

This is proved by an argument similar to the demonstration of Theorem IV., which the student can easily supply.

Theorems I.–VI. were given first, if I mistake not, by Betti, an Italian Mathematician, in the 'Annali, &c.' of Turin and Rome. I am responsible for the terms *derangement* and *model*, as well as for the following terms.

8. *Derived derangements of a group.* The products $GM$ and $MG$ may or may not contain the same substitutions. When they do, *i.e.*, when $GM = MG$, we say that $GM$ is a *derived derangement of G.*

All the groups of Article 2, which are written in two or more columns, as $C$, $g_1$, $H$, $I$, $J$, are composed of a model group $G$, and one or more derived derangements. Thus

$$H_1 = G + MG \text{ is } \begin{array}{ll} 1234 & 1432 \\ 2341 & 4321 \\ 3412 & 3214 \\ 4123, & 2143, \end{array}$$

and

$$H_1 = G + GM \text{ is } \begin{array}{ll} 1234 & 1432 \\ 2341 & 2143 \\ 3412 & 3214 \\ 4123, & 4321, \end{array}$$

where we see that ($M = 1432$) $MG = GM$.

If we take $M' = 3214$, we have

$$M'G = \begin{array}{l} 3214 \\ 2143 \\ 1432 \\ 4321, \end{array} \qquad GM' = \begin{array}{l} 3214 \\ 4321 \\ 1432 \\ 2143. \end{array}$$

In this example we see verifications of Theorems I. and IV.

Perhaps a no less important classification, as we shall come to see, of groups, is given by the distinction and definitions following:

*A model group $H$ is* modular, *if it can be written as a model $G$, followed only by certain derived derangements of $G$.*

*A model group is* non-modular, *when not being a mere group of powers of a substitution, it cannot be written as a model $G$, followed only by derived derangements of $G$.*

No non-modular group exists, if the number of elements be fewer than five.

All the groups of Art. 2 are modular, except only $B$, and the groups of two substitutions, ($K = 2$).

Thus $G'$, though not written as modular, is

$$G' = \begin{array}{ll} 1234 & 2341 \\ 3412, & 4123, \end{array}$$

or is constructible by adding to a model group of the second order a derived derangement. And

$$H' = \begin{array}{llll} 1234 & 2341 & 4321 & 3214 \\ 3412, & 4123, & 2143, & 1432, \end{array}$$

is constructible on the same model of the second order.

It is constructible also on the model $E$, and so are $H'$ and $H''$.

A modular group is, in general, constructible on models of different orders, and often on more than one model of the same order.

9. All the groups of the second order are of the form $1, \theta$, where $\theta^2 = 1$.

The group $B$ is of the form $1, \theta, \theta^2$, where $\theta^3 = 1$.

The groups $G', G'', G'''$ are of the form $1, \theta, \theta^2, \theta^3$, where $\theta^4 = 1$. All these groups are *groups of powers of a substitution.*

The groups $A$ and $B$ are *prime groups.*

*A prime group is a group of the N powers of a substitution made with N elements, when N is any prime number.* It is always a group of the $N^{\text{th}}$ order.

Another classification of groups is that into *transitive* and *intransitive*, which was introduced by Cauchy.

*A group is transitive, if, when it is written in one column, every vertical row contains every element; otherwise it is intransitive.*

The groups $a_1a_2a_3$, $e_1e_2e_3e_4e_5e_6$, $f_1f_2f_3f_4$, $g_1g_2g_3g_4$, having each an element or elements undisturbed, are of course intransitive. The groups $F$, $F'$, $F''$, in which no element is undisturbed, are also intransitive.

The group $E$ is a *grouped group*, and is made by substituting in the auxiliary group $\begin{matrix}1'2'\\2'1'\end{matrix}$, the group $\begin{matrix}12\\21\end{matrix}$ for $1'$, and the group $\begin{matrix}34\\43\end{matrix}$ for $2'$.

The group $F$ is a *woven group*, and is made by weaving together in an obvious manner the substitutions of the group $\begin{matrix}12\\21\end{matrix}$ with those of $\begin{matrix}34\\43\end{matrix}$.

The groups $F'$ and $F''$ are also *woven groups.*

The distinctive characters of grouped and woven groups have been long observed: for the nomenclature, which will be found useful, I am accountable.

(*To be continued.*)

# ON THE LINES OF CURVATURE OF CONICOIDS.

By *A. F. Torry, B.A.*, Scholar of St. John's College, Cambridge.

LET $x$, $y$, $z$ be the coordinates of a point on the conicoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \text{.....................} (1).$$

Along the lines of curvature consecutive normals to the surface intersect, in the point $(\xi, \eta, \zeta)$ suppose: then the equations to the normal are

$$\frac{\xi-x}{\frac{x}{a^2}}=\frac{\eta-y}{\frac{y}{b^2}}=\frac{\zeta-z}{\frac{z}{c^2}},$$

and each fraction $$=\left\{\frac{\rho^2}{\frac{1}{p^2}}\right\}^{\frac{1}{2}}=-p\rho,$$

where $\rho$ is the radius of curvature of the normal section of the surface, whose tangent coincides with that to the line of curvature; $p$ is the perpendicular from the centre on the tangent plane; and the negative sign is taken because $\rho$ is measured in the direction opposite to that of the coordinates: thus

$$\left.\begin{aligned}\xi&=x\left(1-\frac{p\rho}{a^2}\right)\\ \eta&=y\left(1-\frac{p\rho}{b^2}\right)\\ \zeta&=z\left(1-\frac{p\rho}{c^2}\right)\end{aligned}\right\} \text{....................} (2).$$

Since two consecutive normals intersect in the point $(\xi, \eta, \zeta)$ we may differentiate the above equations considering $\xi$, $\eta$, $\zeta$, and $\rho$ constant, writing then $l$, $m$, $n$ for $\frac{dx}{ds}$, $\frac{dy}{ds}$, $\frac{dz}{ds}$, the direction-cosines of the tangent to the line of curvature, we get

$$\left.\begin{aligned} l\left(1-\frac{p\rho}{a^2}\right) &= \frac{x}{a^2}\,\rho\,\frac{dp}{ds} \\ m\left(1-\frac{p\rho}{b^2}\right) &= \frac{y}{b^2}\,\rho\,\frac{dp}{ds} \\ n\left(1-\frac{p\rho}{c^2}\right) &= \frac{z}{c^2}\,\rho\,\frac{dp}{ds} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ (3).$$

These being deduced from the fundamental property of lines of curvature are the differential equations to such lines on the conicoid. We may write them

$$\left.\begin{aligned} l &= \frac{\dfrac{x}{a^2}}{\dfrac{1}{a^2}-\dfrac{1}{p\rho}}\left(-\frac{1}{p}\,\frac{dp}{ds}\right) \\ m &= \frac{\dfrac{y}{b^2}}{\dfrac{1}{b^2}-\dfrac{1}{p\rho}}\left(-\frac{1}{p}\,\frac{dp}{ds}\right) \\ n &= \frac{\dfrac{z}{c^2}}{\dfrac{1}{c^2}-\dfrac{1}{p\rho}}\left(-\frac{1}{p}\,\frac{dp}{ds}\right) \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ (4).$$

By differentiating (1), or from the condition that the normal is at right angles to any tangent, we get

$$\frac{lx}{a^2}+\frac{my}{b^2}+\frac{nz}{c^2}=0 \ldots\ldots\ldots\ldots\ldots\ldots\ (5);$$

therefore, by (4),

$$\frac{\dfrac{x^2}{a^4}}{\dfrac{1}{a^2}-\dfrac{1}{p\rho}}+\frac{\dfrac{y^2}{b^4}}{\dfrac{1}{b^2}-\dfrac{1}{p\rho}}+\frac{\dfrac{z^2}{c^4}}{\dfrac{1}{c^2}-\dfrac{1}{p\rho}}=0 \ldots\ldots\ldots\ (6).$$

Now the values of $u^2$ obtained from the equation

$$\frac{\lambda^2}{A-u^2}+\frac{\mu^2}{B-u^2}+\frac{\nu^2}{C-u^2}=0$$

are the squares of the reciprocals of the semi-axes of the section of the conicoid $Ax^2+By^2+Cz^2=1$, by the plane

$\lambda x + \mu y + \nu z = 0$; and the direction-cosines of an axis corresponding to one of these values are proportional to

$$\frac{\lambda}{A-u^2},\ \frac{\mu}{B-u^2},\ \frac{\nu}{C-u^2}.$$

Comparing these with (6) and (4), we conclude

I. That the values of $p\rho$ are the squares of the semi-axes of the central section of the given conicoid by a plane parallel to the tangent plane at the point $(x, y, z)$.

II. That the directions of the lines of curvature are the same as those of the axes of this section.

PROP. *Along a line of curvature $pD$ is constant, $D$ being* the semi-axis parallel to the tangent to the line of curvature.

Differentiating equation (5)

$$\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}+\frac{x\frac{dl}{ds}}{a^2}+\frac{y\frac{dm}{ds}}{b^2}+\frac{z\frac{dn}{ds}}{c^2}=0,$$

or
$$\frac{1}{D^2}+\frac{x\frac{dl}{ds}}{a^2}+\frac{y\frac{dm}{ds}}{b^2}+\frac{z\frac{dn}{ds}}{c^2}=0 \text{...........}(7).$$

Multiply equations (3) by $\frac{dl}{ds}, \frac{dm}{ds}, \frac{dn}{ds}$, respectively, and add, then

$$l\frac{dl}{ds}+m\frac{dm}{ds}+n\frac{dn}{ds}-\left\{\frac{l\frac{dl}{ds}}{a^2}+\frac{m\frac{dm}{ds}}{b^2}+\frac{n\frac{dn}{ds}}{c^2}\right\}p\rho$$
$$=\left\{\frac{x\frac{dl}{ds}}{a^2}+\frac{y\frac{dm}{ds}}{b^2}+\frac{z\frac{dn}{ds}}{c^2}\right\}\rho\frac{dp}{ds},$$

or, by (7),

$$-\frac{1}{2}\frac{d}{ds}\left(\frac{1}{D^2}\right)p=-\frac{1}{D^2}\frac{dp}{ds},$$
$$\frac{d}{ds}\left(\frac{1}{D^2}\right)p^2=\frac{1}{D^2}\frac{d}{ds}(p^2),$$

whence
$$p^2D^2=\text{constant}.$$

Q. E. D.

# ON CONES, &c.

By *A. F. Torry.*

1. The diametral properties of cones deserve, from their simplicity, a separate investigation; to apply to them the general formulæ for conicoids is much like finding the bisectors of two straight lines in plane geometry by considering them as the limiting form of an hyperbola whose axes are required.

2. It will be necessary first to call attention to a few elementary properties of the cone.

*All polar planes pass through the vertex.*

This is true of tangent planes and of their intersections, and therefore of polar planes which are the loci of such intersections.

*The polar planes of all points in the same straight line through the vertex coincide.*

Since such a line is a diameter the polar planes are all parallel, and thus by what has been shown above coincide.

These properties may also be proved by the following consideration. Take any diameter: the polar plane of its extremity at the vertex is a tangent at that point; that of the other extremity at infinity is parallel to this tangent plane and passes through the centre, *i.e.* the vertex; these then, and the polar planes of intermediate points on the same diameter coincide.

Instead of the polar plane of a point it will be usually convenient to speak of the diametral plane of the line joining that point to the vertex.

3. *To find the plane conjugate to a given diameter.*

Take any convenient section of the cone, and let $P$ (fig. 20) be the point in which the diameter $VP$ meets it, $QR$ the polar line with respect to it of $P$: then $VQR$ is the diametral plane of $VP$.

For the polar plane of $P$ passes through $V$ the vertex (Art. 2), and contains the straight line $QR$.

4. *If a cone circumscribe a conicoid, any point in the plane of contact has the same polar plane with respect to the conicoid and circumscribing cone.*

Such a point lying in the polar plane of the vertex has for its polar with respect to the conicoid the plane passing through the vertex and containing its polar line with respect to the curve of contact: the same construction holds for the cone by Art. 3.

5. PROP. *The axes of the circumscribing cone in* Art. 4 *are the normals to the confocal conicoids passing through its vertex.*

Let $V$ be the vertex of the cone, $p$, $q$, $r$ its coordinates;

$$\frac{x^2}{\alpha}+\frac{y^2}{\beta}+\frac{z^2}{\gamma}=1 \;\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$\frac{xp}{\alpha}+\frac{yq}{\beta}+\frac{zr}{\gamma}=1 \;\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$\frac{x^2}{\alpha+\lambda}+\frac{y^2}{\beta+\lambda}+\frac{z^2}{\gamma+\lambda}=1 \;\ldots\ldots\ldots\ldots (3),$$

the equations to the given conicoid, the polar plane $PQR$ of $V$, and a confocal conicoid through $V$. Also,

$$\frac{p^2}{\alpha+\lambda}+\frac{q^2}{\beta+\lambda}+\frac{r^2}{\gamma+\lambda}=1 \;\ldots\ldots\ldots\ldots (4).$$

The equations to $VP$, the normal to (3) at $V$, are

$$\frac{x-p}{\dfrac{p}{\alpha+\lambda}}=\frac{y-q}{\dfrac{q}{\beta+\lambda}}=\frac{z-r}{\dfrac{r}{\gamma+\lambda}}.$$

Combining these with (2) we get, for the coordinates of $P$,

$$p\frac{\alpha}{\alpha+\lambda},\quad q\frac{\beta}{\beta+\lambda},\quad r\frac{\gamma}{\gamma+\lambda}.$$

The polar plane of this point with respect to (1), *i.e.* the diametral plane of $VP$ with respect to the cone (Art. 4) has for its equation

$$\frac{xp}{\alpha+\lambda}+\frac{yq}{\beta+\lambda}+\frac{zr}{\gamma+\lambda}=1,$$

but this is the equation to the tangent to (3), and is perpendicular to $VP$ the normal; therefore $VP$ (having its diametral plane perpendicular to it) is an axis of the cone.

The three values of $\lambda$ determined by (4) give three confocal conicoids whose normals are the axes required.

Cor. If these axes be $VP$, $VQ$, $VR$, their diametral planes $VQR$, $VRP$, $VPQ$ are, by Art. 4, the polar planes of $P$, $Q$, $R$ with respect to the conicoid; also $PQR$ is the polar plane of $V$, in other words, *the tetrahedron VPQR is self-conjugate with respect to the circumscribed conicoid.* (Senate-House Examination, Jan. 15, 1862).

6. The normals to two of the conicoids are tangents to the lines of curvature on the third; hence the three axes of the circumscribing cone are the normals and tangents to the lines of curvature of a confocal conicoid through its vertex. When the vertex is an umbilicus on one of these surfaces the directions of the lines of curvature are indeterminate and *the cone becomes one of revolution.* It may be asked what becomes of the normals to the other two conicoids and of those surfaces themselves. We will consider this point hereafter.

7. Prop. *If a cone be described with any point of a conicoid as its vertex and a focal as its base, the normal to the conicoid at the vertex is an axis of the cone.* (Frost and Wolstenholme, *Solid Geometry*, Art. 342).

The normal and the tangent plane at any point meet either of the principal planes in a point and straight line respectively, the latter being the polar of the former with respect to the focal in that plane; and thus, by Art. 3, the normal is at right angles to its diametral plane, or *is an axis of the cone.*

The property quoted above is called by some writers the fundamental property of focal curves: it may be very easily deduced from their equations.

8. The remarks on the previous proposition apply equally to this, indeed this is a particular case of the preceding. For let

$$\frac{x^2}{\alpha-\lambda}+\frac{y^2}{\beta-\lambda}+\frac{z^2}{\gamma-\lambda}=1$$

be the equation to a conicoid confocal with

$$\frac{x^2}{\alpha}+\frac{y^2}{\beta}+\frac{z^2}{\gamma}=1,$$

then if we make $\lambda=\gamma$, the cone in the second proposition circumscribes this conicoid which has, so to speak, *degenerated into a focal curve.*

9. *On the confocal conicoids.*

The equation to a conicoid confocal with

$$\frac{x^2}{\alpha}+\frac{y^2}{\beta}+\frac{z^2}{\gamma}=1 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and passing through a point $(p, q, r)$ on it, is

$$\frac{x^2}{\alpha-\lambda}+\frac{y^2}{\beta-\lambda}+\frac{z^2}{\gamma-\lambda}=1 \quad\ldots\ldots\ldots\ldots\ldots (2),$$

where
$$\frac{p^2}{\alpha-\lambda}+\frac{q^2}{\beta-\lambda}+\frac{r^2}{\gamma-\lambda}=1,$$

and
$$\frac{p^2}{\alpha}+\frac{q^2}{\beta}+\frac{r^2}{\gamma}=1.$$

Subtracting the last two equations we get the same quadratic for determining $\lambda$ as we have for the axes of the central section parallel to the tangent plane to (1) at the point $(p, q, r)$; also the expressions for the directions of the *normal* to (2) and of the corresponding axis are the same. (Compare Equations 4, 6 *et seq.* of the preceding paper).

From this we conclude:

I. That as we proceed along the line of intersection of (1) and (2) [one line of curvature on (1)], the semi-diameter $D'$ parallel to the normal to (2) [the tangent to the other line of curvature] is of constant length $\{\sqrt{(\lambda)}\}$; and since, with the usual notation, $pDD'=abc$ we get the theorem that along a line of curvature $pD=$ constant.

II. That since the diameters of a central section in an ellipsoid are one greater and the other less than the mean axis, as may be seen from the equation which determines them, the two confocal conicoids (2) to an ellipsoid (1) are hyperboloids of one and two sheets respectively, degenerating when the point $(p, q, r)$ is an umbilicus into the same curve, *the umbilical focal conic of the ellipsoid.*

10. We have spoken of a conicoid as degenerating into a curve when one of its axes decreases indefinitely and ultimately vanishes. This is not strictly accurate, but corresponding to small variations of this axis the surface changes its character entirely; for instance, in the ellipsoid as the axis passes through the value zero from real to imaginary existence, the surface changes into an hyperboloid of one sheet, the connecting link between the two surfaces being the curve into which we have said the surface degenerates.

Again, just as the indeterminateness at an umbilicus of the directions of lines of curvature is explained by saying that for points indefinitely near it, the lines of curvature have all possible directions, so we may explain the difficulty alluded to in Art. 6. Or we may simply speak of the surfaces as having degenerated into the same curve, and the directions of their normals thereby become indeterminate.

11. It is easily seen that the dirigent focal of an ellipsoid is the umbilical focal of the confocal hyperboloids of two sheets.

More generally the three focals (one of which is imaginary and corresponds to the imaginary umbilici of the hyperboloids of one sheet) of any conicoid, are respectively "the umbilical focals of the three systems of confocal conicoids," or they constitute "the locus of the umbilici of all conicoids confocal with the given one."

The theorem demonstrated in Art. 6 may therefore be enunciated thus: *If with any point in one of the focal curves of such a system of confocal conicoids for vertex, we describe a cone circumscribing any surface of the system, or having one of the other focals for its base, the cone will be one of revolution.*

# NOTE ON THE RECTANGULAR HYPERBOLA.

By *A. Cayley, M.A.*

Every conic which passes through the points of intersection of two rectangular hyperbolas is a rectangular hyperbola. In fact, if a conic be referred to rectangular axes, the condition that it may be a rectangular hyperbola is *Coeff. of* $x^2 = -$ *Coeff. of* $y^2$. Hence if $U$, $V$ be any two quadratic functions of $x$, $y$, and if $\lambda$ be a constant, the condition in question, being satisfied for each of the functions $U$, $V$, is also satisfied for the function $U+\lambda V$; and the equation of any conic through the points of intersection of the conics $U=0$, $V=0$, is $U+\lambda V=0$; which proves the theorem in question.

In particular, if from two of the angles of a triangle perpendiculars are let fall on the opposite sides, and if the point of intersection of the perpendiculars and the third angle be joined; then since the first side and the perpendicular upon it are a rectangular hyperbola, and the second side and the perpendicular upon it are a rectangular hyperbola; the third side and the joining line must be a rectangular hyperbola; that is, these two lines must be at right angles to each other. We have thus the well known theorem that the perpendiculars let fall from the angles of a triangle on the opposite sides meet in a point.

The theorem as to the hyperbolas is a particular case of the theorem, that three conics which pass through the same four points are met by any line whatever in six points forming a system in involution. In fact, a rectangular hyperbola is a conic meeting the line at infinity in two points harmonically related to the circular points at infinity; hence two of the conics being rectangular hyperbolas, the foci of the involution are the circular points at infinity; hence these points and the points in which the line at infinity meets the third conic are harmonically related to each other; that is, the third conic is a rectangular hyperbola.

# QUATERNIONS.

By *P. G. Tait, M.A.*

THE following paper contains the substance of a few elementary Lectures delivered in 1858, solely to explain the nature of the new Calculus. The present article accordingly does not pretend to any logical completeness. Much will be assumed, especially where the proofs, though easy and direct, require more space than can be allowed in the present form of publication. The Inventor has laid, in more ways than one, the rigorous foundations of his Calculus, and the present article will have served its purpose if it give a sufficient outline of the subject to induce the reader to study it in Sir William Hamilton's works. A subsequent paper or two will be devoted to examples of the power and simplicity of the method in its applications especially to Physics and Geometry of Three Dimensions.

For the last century and a half the possibility of interpreting the square root of negative unity as indicating *direction*, and not *length*, has been a favourite subject of speculation with mathematicians.

Wallis, in the end of the 17th century, proposed to represent the impossible roots of a quadratic by going *out of* the line on which, if real, they would have been laid off, and his construction is equivalent to the consideration of $\sqrt{(-1)}$ as a directed unit perpendicular to the line on which real numbers are measured.

De Moivre's Theorem very strongly hints at a similar interpretation. Thus, the expression $\{\cos\alpha + \sqrt{(-1)}\sin\alpha\}^n$ is equal to $1$, $\sqrt{(-1)}$, $-1$, $-\sqrt{(-1)}$, $1$ in succession, when $\alpha = \frac{\pi}{2m}$ and $n$ takes the values $0$, $m$, $2m$, $3m$, $4m$. Following out this reasoning it would seem that $\cos\alpha + \sqrt{(-1)}\sin\alpha$ may be regarded as a factor which *turns* a line, without altering its length, through an angle $\alpha$. The first term of the product is real and in the direction of the line, the second is evidently perpendicular to it.

Further extensions were made by Argand and Warren in the present century. These involved the consideration of the representation of the product of two directed lines as another such line, but simplicity was not gained by their methods, as the fearful array of radicals in Warren's Treatise sufficiently proves.

But, beyond this, few attempts were made, or at least recorded, in earlier times, to extend the principle to space of three dimensions; and, though many such have been made within the last thirty years, none, with the single exception of Sir William Hamilton's, of which it is our object to explain the nature, have resulted in simple, practical methods; all, however ingenious, seeming to lead at once to results of fearful complexity. For a lucid and most impartial statement of the claims of his predecessors in this field we refer to the Preface to Sir William Hamilton's *Lectures on Quaternions.*

It was reserved for him to discover the use of $\sqrt{(-1)}$ as a geometric *reality*, tied down to *no particular direction in space*, and this use was the foundation of the singularly elegant, yet enormously powerful Calculus of Quaternions. While all other schemes for the use of $\sqrt{(-1)}$ as indicating direction made one direction in space expressible by real numbers, all the others being imaginaries of some kind, and while thereby a heterogeneousness was introduced into the methods, Sir William Hamilton makes *all* directions *equally imaginary*, or rather *equally real*, thereby ensuring to his Calculus the power of dealing with space indifferently in all directions. In fact, as we shall see, the Quaternion Method is independent of axes or any supposed directions in space, and takes its reference lines solely from the problem it is applied to. In order to explain it in a very elementary manner, however, it is best to begin by assimilating it as closely as we can to the ordinary Cartesian method, or Geometry of Three Dimensions. We shall find in a little, that it is capable of soaring above these entirely, after having used them for its establishment, and indeed, as the Inventor's works amply prove, it can be established *ab initio* without any reference to them whatever. Since our object is to explain, as simply as possible, the principles and nature of the method, we shall keep to the former track as avoiding metaphysical subtleties, and as giving, though not perhaps in the neatest form, an introduction to a knowledge of the subject. For this purpose it will be necessary to refer carefully to our elementary Geometry of Three Dimensions.

I. Suppose we have two points $A$ and $B$ in *space*, and suppose $A$ given, on how many numbers does $B$'s relative position depend?

If we refer to $x$, $y$, $z$ coordinates (rectangular or not) we find that the data required are the excesses of $B$'s three coordinates over those of $A$. Hence *three* numbers are required.

Or we may take polar coordinates. To define the moon's position with respect to the earth we must have its Geocentric Latitude and Longitude, *or* its Right Ascension and Declination, and, in addition, its distance or radius-vector. *Three* again.

Here it is to be carefully noticed that nothing has been said of the *actual* coordinates of either $A$ or $B$, or of the earth and moon, in space; it is only the *relative* coordinates that are required.

Hence a line as $AB$ contains implicitly *three* numbers, and all lines parallel and equal to $AB$ depend in the same way upon the same three. Hence, *all lines which are equal and parallel may be represented by a common symbol, and that symbol contains three distinct numbers.* In this sense a line is called a VECTOR.

We may here remark, once for all, that in establishing a new Calculus, we are at liberty to give any definitions we choose to the meanings of our symbols, provided that no two of these interfere with, or contradict, each other, and in doing so in Quaternions *simplicity* and (so to speak) *naturalness* were the Inventor's aim.

Let $AB$ be represented by $\alpha$, we know that $\alpha$ depends on *three* separate numbers. Now if $CD$ be equal in length to $AB$ and if these lines be parallel, we have evidently $CD = AB = \alpha$, where it will be seen that the sign of *equality* between vectors contains implicitly *equality in length* and *parallelism in direction.* So far we have *extended* the meaning of an algebraic symbol.

II. We must now define + (and the meaning of − will follow) in the new Calculus. Let $A$, $B$, $C$ be any three points and (with the above meaning of =) let

$$AB = \alpha, \quad BC = \beta, \quad AC = \gamma.$$

If we define + by the equation

$$\alpha + \beta = \gamma,$$

or

$$AB + BC = AC,$$

we contradict nothing that precedes, but we at once introduce the idea that *vectors are to be compounded, in direction and magnitude, like forces.* This may be seen in another way if we remember that by *adding* the differences of $x$, $y$, $z$ coordinates of $A$ and $B$, to those of $B$ and $C$, we get those of $A$ and $C$.

But we also see that if $C$ and $A$ coincide (and $C$ may be *any* point) $AC = 0$. Hence the above relation may be written, in this case,

$$AB + BA = 0,$$

or, introducing, and by the same act defining, the symbol −,

$$BA = -AB.$$

Hence, *the symbol −, applied to a vector, simply shows that it is to be taken in the opposite direction.*

And this is consistent with all that precedes, for instance

$$AB + BC = AC,$$

and

$$AB = AC - BC,$$

or

$$= AC + CB,$$

are evidently but different expressions of the same truth. But we cannot afford to dwell long upon these minor, though important, matters. We may merely mention that Sir W. Hamilton has constructed upon them an admirable theory of "*Vectors*," which includes the results of Möbius' *Mechanik des Himmels*, and those of Grassmann's *Ausdehnungslehre* as particular cases.

III. We must now consider points in which the Quaternion Calculus differs entirely from any previous one, and here we shall get an idea of *what a Quaternion is.* We allude to the new use of the symbols of multiplication and division. It is perhaps simpler to commence with the latter, though in reality we shall mix up the two ideas, as our object is not systematic proof but illustration and explanation.

Let us consider any two vectors, as $AB$ and $DE$; or, if $AC=DE$ (in the Quaternion sense of =) the two $AB$ and $AC$. On how many numbers does the ratio of $AC$ to $AB$ depend? This may be investigated in several ways, but the following, if not the most simple, is at least the most useful as we advance, and, besides, exhibits one of the distinguishing features of the Calculus.

Suppose $AB$ to be changed into $AC$ by the operations of (1st) extending it or contracting it till its length is equal to that of $AC$, and (2nd) turning it about $A$ in the plane of the two lines till it coincides with $AC$. In the first operation *one* number is required, viz. the ratio of the lengths of the two lines. In the second *three* numbers are required, viz. *two* of the three direction-cosines of the normal to the plane in which the rotation is effected, and *one* for the angle of rotation. *The multiplier*, then, *which is required to change one vector into another*, *depends on four distinct numbers*, and is thus a QUATERNION.

We have said that the method we have employed is the most useful; in fact it will be convenient in certain cases to consider a Quaternion as a machine, which when it acts on a vector turns it about a definite axis, through a definite angle, and increases its length in a definite ratio. (It is to be carefully noticed that the only case as yet considered is that in which the axis or handle of the Quaternion is perpendicular to the line).

Hence if $$\beta \div \alpha = \frac{\beta}{\alpha} = q,$$

we have $$\beta = \frac{\beta}{\alpha} \times \alpha = q \times \alpha = q\alpha.$$

(Note again here, we have said $q$ acts on $\alpha$, not $\alpha$ on $q$, and we shall find afterwards that this consideration is most essential, as $\alpha q$ is *not* generally $= q\alpha$).

IV. Let us now look a little more closely into matters, and commence with a rectangular system of vectors, which we may call (after Sir William Hamilton) $I$, $J$, $K$. Let their lengths be unity, and their directions those of three rectangular coordinate axes, suppose those of $x$, $y$, $z$ respectively.

$\frac{K}{J}$ is a Quaternion whose axis is that of $x$, we may therefore assume

$$\frac{K}{J} = i,$$

or $$K = iJ,$$

where $i$ operates as a handle in the direction of $x$ positive, and turns a vector perpendicular to it through a right angle in the direction from $y$ to $z$. (We shall keep up the circular order $x$ to $y$, $y$ to $z$, $z$ to $x$, as will easily be seen when we introduce $j$ and $k$ as factors for the $y$ and $z$ axes, analogous to $i$ for the $x$ axis).

Also, evidently, since the turning and stretching operations of III. are independent, if $n$ be any number,

$$i\,(nJ) = nK;$$

and of this a particular case is

$$i\,(-J) = -K.$$

If we were to write this as

$$(-i)\,J = -K,$$

and compare with $$iJ = \ \ K,$$

it would appear that changing the sign of $i$ merely *reverses* the direction of the handle. This gives a first hint of what we will speedily prove, that *$i$ may be considered as the vector $I$.*

Without going so far, we see that consistently with what precedes we may make

$$iK = -J.$$

But since $i$ turns any vector perpendicular to $I$ through a right angle about $I$ as an axis of right-handed rotation,

$$iK = i\,(iJ) = i^2J;$$

where it is to be observed that we assume the ASSOCIATIVE Law of Multiplication.

Hence $$i^2 = j^2 = k^2 = -1.$$

Also we have, evidently, by referring mentally to the figure,

$$j\,(-I) = K.$$

Hence $$ij\,(-I) = iK = -J = k\,(-I),$$

and therefore $$ij = k.$$

In the same way

$$i\,(-J) = -K,$$

$$ji\,(-J) = -jK = -I = kJ = -k\,(-J).$$

Hence $$ji = -k,$$

and therefore $$ij = -ji.$$

The rules of $i$, $j$, $k$, then, considered as handles which impart a *quadrantal* rotation to a vector perpendicular to them, are summed up in

$$i^2 = j^2 = k^2 = -1,$$

$$ij = -ji = k, \text{ \&c., \&c.,}$$

and it is easy to see that for the last set of equations we may substitute the single one

$$ijk = -1,$$

which, with the values of the squares of $i$, $j$, $k$ completes the summary of their properties.

V. Now if we compare the various equations in last section we find such cases as the following:

$$\left.\begin{aligned} ij &= k \\ iJ &= K \end{aligned}\right\}, \text{ \&c., \&c.}$$

From these, which are all found consistent on examination, we see that we may take $i$, $j$, and $k$ as *equivalent* to $I$, $J$, and $K$ respectively.

Hence, *the quotient of two rectangular unit vectors may be represented by a third unit vector perpendicular to both*, and such that the rotation about it from the Divisor vector to the Dividend is right-handed, or in the direction of rotation of the hands of a watch.

VI. We have alluded already to the stretching and turning elements of a Quaternion, and it is convenient occasionally to consider them separately. For this purpose the Inventor has introduced the notation $Tq$ and $Uq$, representing respectively these elements of the Quaternion $q$. They are called the *Tensor* and *Versor* of $q$. It is now easy to see that, if $q$ be a vector, $Tq$ is its length, and $Uq$ an unit-vector in its direction: and if $q$ be a number or *Scalar* (as Sir William Hamilton calls it)

$$Tq = q \text{ (taken } \textit{positively} \text{ if necessary)},$$

$$Uq = \pm 1 \text{ (as } q \text{ is } + \text{ or } -).$$

Hence $q = TqUq$, where $Tq$ is a "*signless*" number, and $Uq$, as we shall soon see, is a *power* (generally fractional) of an unit-vector; the *first* or some *odd* power, if $q$ is itself a vector; and some *even* integral power if $q$ be a number merely.

VII. Again it is easy to see by the addition of vectors that $x\alpha$, where $x$ is a *number*, represents a vector parallel to $\alpha$ and $x$ times its length, and since $AB = -BA$ (whatever points $A$ and $B$ may be) $-x\alpha$ represents a vector $x$ times the length of $\alpha$, whose direction is parallel but *opposite*.

We may therefore now proceed to extend the results of V. to the finding of the quotient of *any* two vectors, remembering always, that at each new assumption we must carefully ascertain whether it interferes with, or contradicts, anything that has been previously assumed or proved. As we have premised, however, that we do not aim at a logically complete exposition of the Calculus, we may now assume the DISTRIBUTIVE law of vector division to hold generally (as it is really found to do).

Let it be required to find the value of the Quaternion $\frac{\beta}{\alpha}$, $\beta$ and $\alpha$ being given vectors.

$\beta$ may by the law of vector addition be decomposed into two, one parallel, the other perpendicular, to $\alpha$. By our late results the former may be denoted by $x\alpha$, where $x$ is some number; and the latter by $\gamma\alpha$, where $\gamma$ is a vector perpendicular to both $\alpha$ and $\beta$. Hence

$$\beta = x\alpha + \gamma\alpha,$$

or

$$\frac{\beta}{\alpha} = x + \gamma.$$

In this form, again, the Quaternion is seen to depend on four numbers, for $x$ is *one* and $\gamma$ contains (being a vector) *three.*

VIII. The same result might easily have been arrived at thus. Every vector can be resolved into three, parallel respectively to $i$, $j$, $k$ whatever may be the directions of the latter. Hence we may take

$$\alpha = xi + yj + zk,$$

$$\beta = x'i + y'j + z'k,$$

and assume

$$\frac{\beta}{\alpha} = \omega + \xi i + \eta j + \zeta k,$$

where $x$, $y$, $z$, $x'$, $y'$, $z'$, $\omega$, $\xi$, $\eta$, $\zeta$ are numbers.

This is evidently possible, since by the rules of $i$, $j$, $k$ the product of two or more of them is either a multiple of one of them or a number.

Therefore we have, multiplying the sides of the last equation into $\alpha$,

$$\beta = (\omega + \xi i + \eta j + \zeta k)\,\alpha,$$

or $$x'i + y'j + z'k = (\omega + \xi i + \eta j + \zeta k)\,(xi + yj + zk).$$

Multiplying out and comparing coefficients, since evidently there are four distinct kinds of terms,

$$\begin{aligned} 0 &= \quad\;\; \xi x + \eta y + \zeta z, \\ x' &= \omega x \qquad + \eta z - \zeta y, \\ y' &= \omega y - \xi z \qquad + \zeta x, \\ z' &= \omega z + \xi y - \eta x, \end{aligned}$$

a system of four linear equations to determine $\omega$, $\xi$, $\eta$, $\zeta$.

From these, at once,

$$xx' + yy' + zz' = \omega\,(x^2 + y^2 + z^2) = \omega\,(T\alpha)^2.$$

Also we find

$$\xi x' + \eta y' + \zeta z' = 0,$$

and therefore

$$\frac{\xi}{yz' - zy'} = \frac{\eta}{zx' - xz'} = \frac{\zeta}{xy' - yx'} = \frac{1}{x^2 + y^2 + z^2} = \frac{1}{(T\alpha)^2}.$$

IX. With the notation of last article, it is easy to see that

(1) $\dfrac{1}{\alpha} = \alpha^{-1} = -\dfrac{xi + yj + zk}{x^2 + y^2 + z^2} = \dfrac{-\alpha}{(T\alpha)^2}.$

(2) $\alpha^2 = (xi + yj + zk)^2 = -(x^2 + y^2 + z^2) = -(T\alpha)^2.$

(3) $\alpha\beta = (xi + yj + zk)\,(x'i + y'j + z'k)$

$\qquad = -(xx' + yy' + zz') + (yz' - zy')\,i + \ldots.$

(4) $\beta\alpha = -(xx' + \ldots\ldots\ldots) - (yz' - zy')\,i - \ldots.$

Here in (3) and (4) we notice that the quaternions $\alpha\beta$ and $\beta\alpha$ have equal numerical parts, while their vector parts differ in sign. Sir William Hamilton expresses the numerical or scalar, and the vector, parts of a Quaternion $q$ by the forms $Sq$ and $Vq$. Hence we see that

(5) $S\alpha\beta = S\beta\alpha.$ (5′) $\alpha\beta + \beta\alpha = 2S\alpha\beta.$

(6) $V\alpha\beta = -V\beta\alpha.$

Also (again assuming the Associative Law of Multiplication, which can be easily verified by substituting the $i$, $j$, $k$ values of $\alpha$ and $\beta$)

$$\beta\alpha . \alpha\beta = \beta\alpha^2\beta = \beta^2\alpha^2,$$

since $\alpha^2$ is a number by (2). That is (by (5) and (6))

$$(S\alpha\beta - V\alpha\beta)(S\alpha\beta + V\alpha\beta) = \alpha^2\beta^2,$$

or $$S^2.\alpha\beta - V^2.\alpha\beta = \alpha^2\beta^2 = (T\alpha)^2 (T\beta)^2 = T^2.\alpha\beta.$$

This in numbers is the well known theorem of algebra

$$(xx' + yy' + zz')^2 + (yz' - zy')^2 + \ldots = (x^2 + y^2 + z^2)(x'^2 + y'^2 + z'^2).$$

X. If $q$ and $r$ be any two Quaternions

$$\begin{aligned} qr &= (Sq + Vq)(Sr + Vr) \\ &= Sq\,Sr + Sq\,Vr + Sr\,Vq + Vq\,Vr, \end{aligned}$$

(assuming the distributive law).

Hence $$Sqr = Sq\,Sr + S\,Vq\,Vr,$$

and $$Srq = Sr\,Sq + S\,Vr\,Vq.$$

Hence, by IX. (5),

(1) $$Sqr = Srq.$$

As a particular instance let $q = \alpha$, $r = \beta\gamma$, and we have

(2) $$S\alpha\beta\gamma = S\beta\gamma\alpha = S\gamma\alpha\beta,$$

the latter being derived by a second application of the process.

But $$\begin{aligned} S\alpha\beta\gamma &= S.\alpha(S\beta\gamma + V\beta\gamma) \\ &= S.\alpha V\beta\gamma \text{ (since } \alpha S\beta\gamma \text{ is a vector)} \\ &= -S.\alpha V\gamma\beta. \end{aligned}$$

(3) Hence $$S\alpha\beta\gamma = -S\alpha\gamma\beta.$$

From (2) and (3) we see that the scalar part of the product of three vectors depends only on the (circular) order in which they are multiplied, and merely changes its *sign* when that order is changed.

(4) Also, evidently,

$$Sa\beta a = S\beta aa = -S\beta\,(Ta)^2 = 0.$$

(5) Similarly $S.aV\beta a = 0 = S.aVa\beta.$

XI. By IX. (3) and (4) we see that

$$\beta\gamma - \gamma\beta = 2V\beta\gamma.$$

Hence $2V.aV\beta\gamma$ (using the dot after the first $V$ to show that it extends to the whole expression, a very useful convention)

$$= V.a\,(\beta\gamma - \gamma\beta)$$
$$= V.(a\beta\gamma + \beta a\gamma - \beta a\gamma - a\gamma\beta)$$
$$= V.(a\beta + \beta a)\,\gamma - V(\beta Sa\gamma + \beta Va\gamma + Sa\gamma.\beta + Va\gamma.\beta)$$
$$= 2\gamma Sa\beta - 2\beta Sa\gamma, \text{ by IX. (5') and (6),}$$

and therefore (1) $V.aV\beta\gamma = \gamma Sa\beta - \beta Sa\gamma,$

a *most important* formula.

(Note, in the expression $Va\gamma.\beta$, above used, the point indicates that $Va\gamma$ multiplies $\beta$: we have no other way of indicating this, as we cannot generally, without altering signs, &c., change the order of vector factors).

It follows immediately that

$$\begin{aligned} (2) \qquad V.a\beta\gamma &= V.aS\beta\gamma + V.aV\beta\gamma \\ &= aS\beta\gamma + V.aV\beta\gamma \\ &= aS\beta\gamma - \beta Sa\gamma + \gamma Sa\beta, \end{aligned}$$

a formula, like (1), to be carefully remembered.

By (1) we see at once that

$$\begin{aligned} (3) \qquad V.Va\beta V\gamma\delta &= -V.V\gamma\delta Va\beta \text{ (by IX. (6))} \\ &= \delta Sa\beta\gamma - \gamma S\delta a\beta \\ &= aS\beta\gamma\delta - \beta S\gamma\delta a \text{ (by XI. (1)).} \end{aligned}$$

Equating the two latter forms, and using the formulæ of X. to modify the scalars,

$$(4) \qquad \delta S\alpha\beta\gamma = \alpha S\beta\gamma\delta + \beta S\gamma\alpha\delta + \gamma S\alpha\beta\delta,$$

another important formula, which gives a vector in terms of three others.

We leave to the reader the proof of a second valuable formula for the same purpose.

$$(5) \qquad \delta S\alpha\beta\gamma = V\alpha\beta\, S\gamma\delta + V\beta\gamma\, S\alpha\delta + V\gamma\alpha\, S\beta\delta.$$

XII. An important consideration, hinted at in VI., leads us to give a definition of a *power* of a Quaternion. We need here consider, however, only the simple case of the Quaternion reducing to a vector.

Let $\alpha$ be an unit-vector perpendicular to $\beta$. Then $\alpha\beta$ is a vector in length equal to $\beta$, but perpendicular to both $\alpha$ and $\beta$; $\alpha^2\beta = -\beta$ is $\beta$ turned through *two* right angles about $\alpha$, $\alpha^3\beta = -\alpha\beta$ is the same turned through three right angles, and so on. Hence it is natural to define $\alpha^m$ as a factor which turns $\beta$ through $m$ right angles about $\alpha$ as an axis. And we may thus express the versor part of any Quaternion as a power of an unit-vector. Thus to take the case in VII., $\frac{\beta}{\alpha} = x + \gamma$, where $\gamma$ is perpendicular to $\beta$ and $\alpha$. Hence dividing out the tensor factor of $\gamma$, we have $\frac{\gamma}{T\gamma} = U\gamma$ as an unit-vector perpendicular to $\beta$ and $\alpha$, and the turning factor or versor of the Quaternion is $(U\gamma)^m$ where $m$ bears to unity the same ratio that the angle between $\beta$ and $\alpha$ bears to a right angle.

XIII. Again, in the same instance from VII., if $\theta$ be the angle between $\alpha$ and $\beta$, it is evident that the length of $x\alpha$ is $T\beta\cos\theta$, and of $\gamma\alpha$, $T\beta\sin\theta$. Hence

$$\begin{aligned}\beta &= T\beta\cos\theta\, U\alpha + T\beta\sin\theta\, U\gamma\, U\alpha \\ &= \frac{T\beta}{T\alpha}(\cos\theta.\alpha + \sin\theta.U\gamma.\alpha),\end{aligned}$$

or $$q = \frac{\beta}{\alpha} = \frac{T\beta}{T\alpha}(\cos\theta + \sin\theta . U\gamma).$$

As $U\gamma$ is an unit-vector, its square is $-1$, and therefore we have the remarkable result, that $N$ being some number, a Quaternion can generally be written

$$q = N\{\cos\theta + \sqrt{(-1)}\,\sin\theta\}.$$

But it is now to be observed that in this Calculus the $\sqrt{(-1)}$ so introduced is a definite vector, and therefore a totally different kind of symbol from that involved in De Moivre's theorem above alluded to. It may merely be mentioned here that the $\sqrt{(-1)}$ of Algebra comes into Quaternions also, but with its algebraic features only, and indicates here as in Coordinate Geometry the impossibility of an intersection, &c. This gives rise to what are called BIQUATERNIONS.

XIV. In the subsequent papers it may be requisite to differentiate a quaternion expression. It is only necessary to remark on this head that in the simple differentiations we will require, the one peculiarity to be noticed is that each factor must be differentiated where it stands, since Quaternion multiplication is not generally commutative. Beyond this there is no new difficulty.

And, in conclusion, it is well to observe that for the successful study of Quaternions it is most essential constantly to work examples, since even the very simplest formulæ, by the infinite variety of transformations of which they are capable, will perplex the learner for a time. But that which forms the chief difficulty to the beginner, is one of the grandest advantages to him who has acquired even a moderate proficiency in the use of the Calculus.

# ON A CERTAIN FORM OF EQUATIONS REPRESENTING A STRAIGHT LINE IN TRILINEAR COORDINATES.

By *William Allen Whitworth, B.A.*, Scholar of St. John's College, Cambridge.

(*Continued from page* 53.)

11*. THE equations of Art. 11,

$$\frac{\lambda}{l-m\cos C-n\cos B}=\frac{\mu}{m-n\cos A-l\cos C}=\frac{\nu}{n-l\cos B-m\cos A}\quad\ldots\ldots\ldots\ldots\ldots(1)$$

involve the relation

$$\lambda\sin A+\mu\sin B+\nu\sin C=0.$$

They therefore express only one other relation.

To find this other, we have, from the first of the equations in (1),

$$l(\mu+\lambda\cos C)-m(\lambda+\mu\cos C)+n(\mu\cos B-\lambda\cos A)=0,$$

whence
$$\frac{l(\mu\cos B-\nu\cos C)}{\sin A}+\frac{m(\nu\cos C-\lambda\cos A)}{\sin B}+\frac{n(\lambda\cos A-\mu\cos B)}{\sin C}=0\ldots\ldots\ldots(2),$$

which therefore expresses the condition necessary and sufficient that $l\alpha+m\beta+n\gamma=0$ should be at right angles to

$$\frac{\alpha-\alpha'}{\lambda}=\frac{\beta-\beta'}{\mu}=\frac{\gamma-\gamma'}{\nu}.$$

We shall presently find this condition useful.

12. *To find the centre of the conic whose equation is*

$$f(\alpha,\beta,\gamma)\equiv u\alpha^2+v\beta^2+w\gamma^2+2u'\beta\gamma+2v'\gamma\alpha+2w'\alpha\beta=0.$$

Let $(\bar\alpha\bar\beta\bar\gamma)$ be the centre, then the equation to any diameter will be

$$\frac{\alpha-\bar\alpha}{\lambda}=\frac{\beta-\bar\beta}{\mu}=\frac{\gamma-\bar\gamma}{\nu}=\rho,$$

and the lengths of its intercepts between the centre and the curve will be the roots of the equation

$$f(\bar{\alpha}+\lambda\rho,\ \bar{\beta}+\mu\rho,\ \bar{\gamma}+\nu\rho)=0.$$

Since the two intercepts are equal, but drawn in opposite directions the roots of this equation must be equal in magnitude but opposite in sign; therefore the coefficient of $\rho$ is zero, that is

$$\lambda(u\bar{\alpha}+v'\bar{\gamma}+w'\bar{\beta})+\mu(v\bar{\beta}+w'\bar{\alpha}+u'\bar{\gamma})+\nu(w\bar{\gamma}+u'\bar{\beta}+v'\bar{\alpha})=0 \ldots (1).$$

But we have also

$$\lambda a+\mu b+\nu c=0.$$

And these two equations hold for all values whatever of the ratios $\lambda : \mu : \nu$; therefore

$$\frac{u\bar{\alpha}+v'\bar{\gamma}+w'\bar{\beta}}{a}=\frac{v\bar{\beta}+w'\bar{\alpha}+u'\bar{\gamma}}{b}=\frac{w\bar{\gamma}'+u'\bar{\beta}+v'\bar{\alpha}}{c} \ldots (2),$$

which equations together with the condition

$$a\bar{\alpha}+b\bar{\beta}+c\bar{\gamma}=2\Delta$$

determine $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ completely.

Adopting the notation of determinants our result will take the form

$$\frac{\bar{\alpha}}{\begin{vmatrix} w', & v', & a \\ v, & u', & b \\ u', & w, & c \end{vmatrix}}=\frac{\bar{\beta}}{\begin{vmatrix} u', & w', & b \\ w, & v', & c \\ v', & u, & a \end{vmatrix}}=\frac{\bar{\gamma}}{\begin{vmatrix} v', & u', & c \\ u, & w', & a \\ w', & v, & b \end{vmatrix}}.$$

Or, with the notation of the differential calculus, the equation (1) may be written

$$\lambda\frac{df(\bar{\alpha},\bar{\beta},\bar{\gamma})}{d\bar{\alpha}}+\mu\frac{df(\bar{\alpha},\bar{\beta},\bar{\gamma})}{d\bar{\beta}}+\nu\frac{df(\bar{\alpha},\bar{\beta},\bar{\gamma})}{d\bar{\gamma}}=0,$$

and the equations (2) which determine the centre become

$$\frac{df}{d\bar{\alpha}}\operatorname{cosec}A=\frac{df}{d\bar{\beta}}\operatorname{cosec}B=\frac{df}{d\bar{\gamma}}\operatorname{cosec}C.$$

13. *To find the length of the diameter in any direction in the conic whose given equation is*

$$f(\alpha,\beta,\gamma)\equiv u\alpha^2+v\beta^2+w\gamma^2+2u'\beta\gamma+2v'\gamma\alpha+2w'\alpha\beta=0.$$

Let the given direction make angles $\sin^{-1}\lambda$, $\sin^{-1}\mu$, $\sin^{-1}\nu$ with the lines of reference, then, if $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ be the coordinates of the centre, the equation to the diameter is

$$\frac{\alpha-\bar{\alpha}}{\lambda}=\frac{\beta-\bar{\beta}}{\mu}=\frac{\gamma-\bar{\gamma}}{\nu}=\rho,$$

and the lengths of the intercepts between $(\bar{\alpha}\bar{\beta}\bar{\gamma})$ and the curve are given by

$$f(\bar{\alpha}+\lambda\rho,\ \bar{\beta}+\mu\rho,\ \bar{\gamma}+\nu\rho)=0.$$

Since $(\bar{\alpha}\bar{\beta}\bar{\gamma})$ is the centre, the roots of this equation must be of equal magnitude and opposite sign; therefore the coefficient of $\rho$ vanishes and the equation becomes

$$f(\bar{\alpha},\bar{\beta},\bar{\gamma})+\rho^2 f(\lambda,\mu,\nu)=0;$$

therefore the length of the semi-diameter is given by

$$\rho^2=-\frac{f(\bar{\alpha},\bar{\beta},\bar{\gamma})}{f(\lambda,\mu,\nu)}.$$

14. *To find the conditions that the general equation of the second degree should represent a circle.*

It is sufficient if we express the conditions that three semi-diameters in different directions should be equal.

Choose as the three diameters those which are parallel to the sides of the triangle of reference; then we must have

$$f(0,c,-b)=f(-c,0,a)=f(b,-a,0),$$

or $vc^2+wb^2-2u'bc=wa^2+uc^2-2v'ca=wb^2+va^2-2u'ab$,

which will be the required conditions.

15. *To find the conditions that the general equation of the second degree should represent a parabola, an ellipse, or a hyperbola.*

The condition that the length of the intercept on the line

$$\frac{\alpha-\alpha'}{\lambda}=\frac{\beta-\beta'}{\mu}=\frac{\gamma-\gamma'}{\nu}=\rho$$

should be infinite is

$$f(\lambda, \mu, \nu) = 0;$$

therefore the conic will have one, two, or no directions in which the radii from a finite point are infinite according as $f(\lambda, \mu, \nu) = 0$ has real and equal, real and unequal, or unreal roots. Eliminating $\nu$ by means of the relation $a\lambda + b\mu + c\nu = 0$, we get

$$\lambda^2 (wa^2 + uc^2 - 2v'ac) + \mu^2 (wb^2 + vc^2 - 2u'bc) \\ + 2\mu\nu (wab - u'ac - v'bc + w'c^2) = 0.$$

Hence the conic is a parabola, hyperbola, or ellipse according as

$$(wa^2 + uc^2 - 2v'ac)(wb^2 + vc^2 - 2u'bc) \begin{smallmatrix} = \\ < \\ > \end{smallmatrix} (wab - u'ac - v'bc + w'c^2)^2,$$

*i.e.* according as

$$a^2 (vw - u'^2) + b^2 (wu - v'^2) + c^2 (uv - w'^2) \\ + 2bc (v'w' - uu') + 2ca (w'u' - vv') + 2ab (u'v' - ww')$$

is zero, negative, or positive.

16. *To find the direction of the axis of the parabola whose given equation is*

$$f(\alpha, \beta, \gamma) \equiv u\alpha^2 + v\beta^2 + w\gamma^2 + 2u'\beta\gamma + 2v'\gamma\alpha + 2w'\alpha\beta = 0.$$

Since the equation

$$\lambda^2 (wa^2 + uc^2 - 2v'ac) + \mu^2 (wb^2 + vc^2 - 2u'bc) \\ + 2\mu\nu (wab - u'ac - v'bc + w'c^2) = 0$$

of Art. 15, has in this case two equal roots, the solution must be

$$\frac{\lambda}{\sqrt{(wb^2 + vc^2 - 2u'bc)}} = \frac{\mu}{\sqrt{(uc^2 + wa^2 - 2v'ca)}},$$

and by symmetry,

$$= \frac{\nu}{\sqrt{(v^2a^2 + ub^2 - 2w'ab)}}.$$

Hence the axis is parallel to the line

$$\frac{\alpha - \alpha'}{\sqrt{(wb^2 + vc^2 - 2u'bc)}} = \frac{\beta - \beta'}{\sqrt{(uc^2 + wa^2 - 2v'ca)}} = \frac{\gamma - \gamma'}{\sqrt{(va^2 + ub^2 - 2w'ab)}}.$$

17. *To find the conditions that the general equat* *second degree may represent a rectangular hyperbola.*

Let $f(\alpha, \beta, \gamma) = 0$ be the equation, then since th totes have double contact at infinity with the cu equation may be written

$$f(\alpha, \beta, \gamma) - \kappa\,(a\alpha + b\beta + c\gamma)^2 = 0;$$

therefore, by Cor. 2 of Art. 7, the asymptotes are angles provided

$$u + v + w - 2u' \cos A - 2v' \cos B - 2c' \cos C$$

$$= \kappa\,(a^2 + b^2 + c^2 - 2bc \cos A - 2ca \cos B - 2ab \cos C) = 0 \text{ i}$$

which is therefore the condition required.

18. *To find the foci of the conic represented by* *equation of the second degree**

$$f(\alpha, \beta, \gamma) \equiv u\alpha^2 + v\beta^2 + w\gamma^2 + 2u'\beta\gamma + 2v'\gamma\alpha + 2w'$$

Let $(\alpha'\beta'\gamma')$ be a focus. Then the focal radius direction $(\lambda, \mu, \nu)$ is given by the equations

$$\frac{\alpha - \alpha'}{\lambda} = \frac{\beta - \beta'}{\mu} = \frac{\gamma - \gamma'}{\nu} = \rho,$$

and the lengths of the two radii vectores in this by the equation

$$\rho^2 f(\lambda, \mu, \nu) + 2\rho\left(\lambda\frac{df}{d\alpha'} + \mu\frac{df}{d\beta'} + \nu\frac{df}{d\gamma'}\right) + f(\alpha', \beta',$$

If $\rho_1$, $\rho_2$ be the roots of this equation, we have

$$\frac{1}{\rho_1} - \frac{1}{\rho_2} = \text{a constant} = \frac{\frac{1}{2}}{\text{latus rectum}};$$

therefore $\left(\lambda\dfrac{df}{d\alpha'} + \mu\dfrac{df}{d\beta'} + \nu\dfrac{df}{d\gamma'}\right)^2 - 4f(\lambda, \mu, \nu) f($

is independent of the direction $(\lambda, \mu, \nu)$, and have the same value if we take this direction successively parallel to the three lines of reference; therefore

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* The method here pursued was St. John's College.

$$\left.\begin{aligned} &\tfrac{1}{4}\left(c\frac{df}{d\beta'}-b\frac{df}{d\gamma'}\right)^2-f(0,c,-b)\,f(\alpha',\beta',\gamma') \\ =&\tfrac{1}{4}\left(a\frac{df}{d\gamma'}-c\frac{df}{d\alpha'}\right)^2-f(-c,0,a)\,f(\alpha',\beta',\gamma') \\ =&\tfrac{1}{4}\left(b\frac{df}{d\alpha'}-a\frac{df}{d\beta'}\right)^2-f(b,-a,0)\,f(\alpha',\beta',\gamma') \end{aligned}\right\}\ldots(1).$$

Suppressing the accents on $\alpha'$, $\beta'$, $\gamma'$, the first member of these equalities may be written

$$\begin{aligned} &b^2\left\{(v'^2-wu)\,\alpha^2+(u'^2-wv)\,\beta^2+2\alpha\beta\,(u'v'-ww')\right\} \\ &+c^2\left\{(w'^2-uv)\,\alpha^2+(u'^2-wv)\,\gamma^2+2\alpha\gamma\,(u'w'-uu')\right\} \\ &-2bc\left\{(w'v'-uu')\,\alpha^2+(vw-u'^2)\,\beta\gamma+(ww'-u'v')\,\gamma\alpha+(vv'-u'w')\,\alpha\beta\right\}, \end{aligned}$$

or adopting Mr. Ferrers' notation,* putting for shortness

$$vw-u'^2=l,\quad wu-v'^2=m,\quad uv-w'^2=n,$$
$$v'w'-uu'=l',\quad w'u'-vv'=m',\quad u'v'-ww'=n',$$

and changing the signs throughout,

$$b^2\,(m\alpha^2+l\beta^2-2n'\alpha\beta)+c^2\,(n\alpha^2+l\gamma^2-2m'\alpha\gamma)$$
$$+2bc\,(l'\alpha^2+l\beta\gamma-n'\gamma\alpha-m'\alpha\beta),$$

or $\alpha^2\,(mb^2+nc^2+2l'bc)+l\,(b\beta+c\gamma)^2-2\,(bn'+cm')\,\alpha\,(b\beta+c\gamma)$.

Hence the equations become

$$\left.\begin{aligned} &\alpha^2(mb^2+nc^2+2l'bc)-2\alpha(b\beta+c\gamma)(bn'+cm')+l(b\beta+c\gamma)^2 \\ =&\beta^2(nc^2+la^2+2m'ca)-2\beta(c\gamma+a\alpha)(cl'+an')+m(c\gamma+a\alpha)^2 \\ =&\gamma^2(la^2+mb^2+2n'ab)-2\gamma(a\alpha+b\beta)(am'+bl')+n(a\alpha+b\beta)^2 \end{aligned}\right\}\ldots(2),$$

and these equations, together with the further condition

$$a\alpha+b\beta+c\gamma=2\Delta\ \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

will determine completely the coordinates of the focus.

It will be found that the equations have generally two real and two unreal solutions, indicating two real and two imaginary foci.

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* *Quarterly Journal of Mathematics*, Vol. IV., p. 236.

By substitution of (3), the system (2) takes the form

$$\left.\begin{aligned} &\alpha^2\{la^2+mb^2+nc^2+2l'bc+2m'ca+2n'ab\}\\ &\qquad\qquad -4\Delta\alpha\{bn'+cm'+al\}+4\Delta^2 l\\ =&\beta^2\{la^2+mb^2+nc^2+2l'bc+2m'ca+2n'ab\}\\ &\qquad\qquad -4\Delta\beta\{cl'+an'+bm\}+4\Delta^2 m\\ =&\gamma^2\{la^2+mb^2+nc^2+2l'bc+2m'ca+2n'ab\}\\ &\qquad\qquad -4\Delta\gamma\{am'+bl'+cn\}+4\Delta^2 n\end{aligned}\right\}\text{......(4)};$$

these equations differ from those obtained by Mr. Ferrers in the *Quarterly Journal*, in the coordinates being the ordinary trilinear instead of triangular coordinates: we do not agree however with the results there obtained with respect to some of the *signs*.

19. These equations become much simpler in some particular cases.

I. Suppose the conic be a circle, then by Art. 13,

$$f(0,c,-b)=f(-c,0,a)=f(b,-a,0),$$

hence the equations (1) of the last article become

$$\left(c\frac{df}{d\beta}-b\frac{df}{d\gamma}\right)^2=\left(a\frac{df}{d\gamma}-c\frac{df}{d\alpha}\right)^2=\left(b\frac{df}{d\alpha}-a\frac{df}{d\beta}\right)^2;$$

therefore $$\frac{1}{a}\frac{df}{d\alpha}=\frac{1}{b}\frac{df}{d\beta}=\frac{1}{c}\frac{df}{d\gamma},$$

or the centre is a focus.

II. Suppose the conic be a parabola, then, by Art. (15),

$$la^2+mb^2+nc^2+2l'bc+2m'ca+2n'ab=0.$$

Hence, by equations (4), one of the foci is a real point at infinity, and the other is given by

$$\begin{aligned}\alpha(bn'+cm'+al)-l\Delta&=\beta(cl'+an'+bm)-m\Delta\\ &=\gamma(am'+bl'+cn)-n\Delta.\end{aligned}$$

III. Suppose it be the circumscribed conic

$$\lambda\beta\gamma+\mu\gamma\alpha+\nu\alpha\beta=0,$$

then the equations (4) become

$$\begin{aligned}&(\lambda^2a^2+\mu^2b^2+\nu^2c^2-2\mu\nu bc-2\nu\lambda ca-2\lambda\mu ab)\,\alpha^2\\&\qquad\qquad+4\Delta\lambda\alpha\,(\mu b+\nu c-\lambda a)+4\Delta^2\lambda^2\\&=(\lambda^2a^2+\mu^2b^2+\nu^2c^2-2\mu\nu bc-2\nu\lambda ca-2\lambda\mu ab)\,\beta^2\\&\qquad\qquad+4\Delta\mu\beta\,(\nu c+\lambda a-\mu b)+4\Delta^2\mu^2\\&=(\lambda^2a^2+\mu^2b^2+\nu^2c^2-2\mu\nu bc-2\nu\lambda ca-2\lambda\mu ab)\,\gamma^2\\&\qquad\qquad+4\Delta\nu\gamma\,(\lambda a+\mu b-\nu c)+4\Delta^2\nu^2,\end{aligned}$$

a very symmetrical system.

IV. Suppose it be the inscribed conic

$$\lambda^2\alpha^2+\mu^2\beta^2+\nu^2\gamma^2-2\mu\nu\beta\gamma-2\nu\lambda\gamma\alpha-2\lambda\mu\alpha\beta=0.$$

In this case the equations become

$$\begin{aligned}&\alpha^2\,(\lambda bc+\mu ca+\nu ab)-2\alpha\Delta\,(b\nu+c\mu)\\&=\beta^2\,(\lambda bc+\mu ca+\nu ab)-2\beta\Delta\,(c\lambda+a\nu)\\&=\gamma^2\,(\lambda bc+\mu ca+\nu ab)-2\gamma\Delta\,(a\mu+b\lambda).\end{aligned}$$

But in consequence of the property of all conics that the product of the perpendiculars from the foci on any tangent is constant, this case admits of a simple independent investigation, thus:

If $(\alpha_1\beta_1\gamma_1)$, $(\alpha_2\beta_2\gamma_2)$ be the two foci, $(\bar{\alpha}\bar{\beta}\bar{\gamma})$ the centre, then

$$\alpha_2=\frac{\kappa^2}{\alpha_1},\quad \beta_2=\frac{\kappa^2}{\beta_1},\quad \gamma_2=\frac{\kappa^2}{\gamma_1},$$

$\kappa$ being a constant, equal in fact to the semi-minor axis; therefore

$$\kappa^2\left(\frac{a}{\alpha_1}+\frac{b}{\beta_1}+\frac{c}{\gamma_1}\right)=a\alpha_2+b\beta_2+c\gamma_2=a\alpha_1+b\beta_1+c\gamma_1,$$

whence
$$\kappa^2=\frac{a\alpha_1+b\beta_1+c\gamma_1}{\dfrac{a}{\alpha_1}+\dfrac{b}{\beta_1}+\dfrac{c}{\gamma_1}}.$$

Now the centre is given by

$$\frac{a}{\lambda}\,(b\bar{\beta}+c\bar{\gamma}-a\bar{\alpha})=\frac{b}{\mu}\,(c\bar{\gamma}+a\bar{\alpha}-b\bar{\beta})=\frac{c}{\nu}\,(a\bar{\alpha}+b\bar{\beta}-c\bar{\gamma}).$$

And $\quad 2\bar{\alpha}=\alpha_1+\dfrac{\kappa^2}{\alpha_1},\quad 2\bar{\beta}=\beta_1+\dfrac{\kappa^2}{\beta_1},\quad 2\bar{\gamma}=\gamma_1+\dfrac{\kappa^2}{\gamma_1};$

therefore $$b\bar{\beta}+c\bar{\gamma}-a\bar{\alpha}=b\beta_1+c\gamma_1-\kappa^2\frac{a}{\alpha_1}.$$

Hence the focus is given by

$$\frac{a}{\lambda}\left(b\beta_1+c\gamma_1-\kappa^2\frac{a}{\alpha_1}\right)=\frac{b}{\mu}\left(c\gamma_1+a\alpha_1-\kappa^2\frac{b}{\beta_1}\right)=\frac{c}{\nu}\left(a\alpha_1+b\beta_1-\kappa^2\frac{c}{\gamma_1}\right),$$

or substituting the value of $\kappa$,

$$\frac{a}{\lambda}\left\{(b\beta_1+c\gamma_1)\left(\frac{b}{\beta_1}+\frac{c}{\gamma_1}\right)-a^2\right\}=\frac{b}{\mu}\left\{(c\gamma_1+a\alpha_1)\left(\frac{c}{\gamma_1}+\frac{a}{\alpha_1}\right)-b^2\right\}$$
$$=\frac{c}{\nu}\left\{(a\alpha_1+b\beta_1)\left(\frac{a}{\alpha_1}+\frac{b}{\beta_1}\right)-c^2\right\},$$

or $$\frac{\frac{\beta_1}{\gamma_1}+\frac{\gamma_1}{\beta_1}+2\cos A}{\lambda}=\frac{\frac{\gamma_1}{\alpha_1}+\frac{\alpha_1}{\gamma_1}+2\cos B}{\mu}=\frac{\frac{\alpha_1}{\beta_1}+\frac{\beta_1}{\alpha_1}+2\cos C}{\nu}.$$

## CONJUGATE DIAMETERS.

20. *To find the equation to the diameter bisecting chords whose direction-sines are* $\lambda, \mu, \nu$, *in the conic* $f(\alpha, \beta, \gamma)=0$.

Let $$\frac{\alpha-\alpha'}{\lambda}=\frac{\beta-\beta'}{\mu}=\frac{\gamma-\gamma'}{\nu}=\rho$$

be one of the chords, and $(\alpha'\beta'\gamma')$ its middle point, then the equation

$$\rho^2 f(\lambda, \mu, \nu)+\rho\left(\alpha'\frac{df}{d\lambda}+\beta'\frac{df}{d\mu}+\gamma'\frac{df}{d\nu}\right)+f(\alpha', \beta', \gamma')=0$$

must have its roots equal in magnitude and of opposite signs; therefore

$$\alpha'\frac{df}{d\lambda}+\beta'\frac{df}{d\mu}+\gamma'\frac{df}{d\nu}=0.$$

This is a relation among the coordinates of the middle point of any of the chords. Hence suppressing the accents,

$$\alpha\frac{df}{d\lambda}+\beta\frac{df}{d\mu}+\gamma\frac{df}{d\nu}=0$$

is the equation to the locus of the middle points, that is, to the diameter conjugate to the diameter

$$\frac{\alpha-\bar{\alpha}}{\lambda}=\frac{\beta-\bar{\beta}}{\mu}=\frac{\gamma-\bar{\gamma}}{\nu}.$$

21. The diameter

$$\frac{\alpha-\bar{\alpha}}{\lambda}=\frac{\beta-\bar{\beta}}{\mu}=\frac{\gamma-\bar{\gamma}}{\nu}$$

will be at right angles to its conjugate, provided

$$\frac{df}{d\lambda}\,\frac{\mu\cos B-\nu\cos C}{a}+\frac{df}{d\mu}\,\frac{\nu\cos C-\lambda\cos A}{b}+\frac{df}{d\nu}\,\frac{\lambda\cos A-\mu\cos B}{c}=0,$$

by equation (2) of Art. 11.

Hence this equation, together with the equation

$$a\lambda+b\mu+c\nu=0,$$

determines the ratios $\lambda:\mu:\nu$, so that the equations

$$\frac{\alpha-\bar{\alpha}}{\lambda}=\frac{\beta-\bar{\beta}}{\mu}=\frac{\gamma-\bar{\gamma}}{\nu}=\rho$$

may represent an *axis* of the conic.

By substitution of $a\lambda+b\mu+c\nu=0$ the above quadratic may be reduced to the form

$$\begin{vmatrix} u-v'\sec B-w'\sec C, & v-w'\sec C-u'\sec A, & w-u'\sec A-v'\sec B \\ \sin 2A, & \sin 2B, & \sin 2C \\ \dfrac{\sin A}{\alpha-\bar{\alpha}}, & \dfrac{\sin B}{\beta-\bar{\beta}}, & \dfrac{\sin C}{\gamma-\bar{\gamma}} \end{vmatrix}=0.$$

We proceed to give an independent investigation of this equation.

22. *To find the equations to the axes of the conic represented by the general equation of the second degree.*

Let
$$\frac{\alpha-\bar{\alpha}}{\lambda}=\frac{\beta-\bar{\beta}}{\mu}=\frac{\gamma-\bar{\gamma}}{\nu}=\rho$$

be the equations to an axis, then by Art. 13, $\lambda$, $\mu$, $\nu$ must have such values as will make $f(\lambda, \mu, \nu)$ assume a maximum or minimum value.

Now

$$f(\lambda,\mu,\nu)\equiv u\lambda^2+v\mu^2+w\nu^2+2u'\mu\nu+2v'\nu\lambda+2w'\lambda\mu,$$

or, in virtue of equations (iii) of Art. 3,

$$\equiv u\lambda^2 + v\mu^2 + w\nu^2 - u' \sec A\,(\mu^2+\nu^2) - v' \sec B\,(\nu^2+\lambda^2)$$
$$- w' \sec C\,(\lambda^2+\mu^2) + \text{constant terms},$$
$$\equiv \lambda^2\,(u - v' \sec B - w' \sec C) + \mu^2\,(v - w' \sec C - u' \sec A)$$
$$+ \nu^2\,(w - u' \sec A - v' \sec B) + \text{constant terms},$$

hence we must have

$$0 = \lambda\delta\lambda\,(u - v' \sec B - w' \sec C) + \mu\delta\mu\,(v - w' \sec C - u' \sec A)$$
$$+ \nu\delta\nu\,(w - u' \sec A - v' \sec B).$$

Also, by equation (iv.) of Art. 4,

$$0 = \lambda\delta\lambda \sin 2A + \mu\delta\mu \sin 2B + \nu\delta\nu \sin 2C.$$

Also $$a\delta\lambda + b\delta\mu + c\delta\nu = 0.$$

Whence eliminating $\delta\lambda$, $\delta\mu$, $\delta\nu$, we get

$$\begin{vmatrix} u - v'\sec B - w'\sec C, & v - w'\sec C - u'\sec A, & w - u'\sec A - v'\sec B \\ \sin 2A, & \sin 2B, & \sec 2C \\ \dfrac{\sin A}{\lambda}, & \dfrac{\sin B}{\mu}, & \dfrac{\sin C}{\nu} \end{vmatrix} = 0;$$

and therefore the axes are given by the quadratic equation

$$\begin{vmatrix} u - v'\sec B - w'\sec C, & v - w'\sec C - u'\sec A, & w - u'\sec A - v'\sec B \\ \sin 2A, & \sin 2B, & \sin 2C \\ \dfrac{\sin A}{\alpha - \bar{\alpha}}, & \dfrac{\sin B}{\beta - \bar{\beta}}, & \dfrac{\sin C}{\gamma - \bar{\gamma}} \end{vmatrix} = 0,$$

the same which was otherwise deduced in the last article.

23. *To determine the conditions that a line, given by an equation* $l\alpha + m\beta + n\gamma = 0$ *of the ordinary form, should be an axis of the conic,*

$$f(\alpha, \beta, \gamma) \equiv u\alpha^2 + v\beta^2 + w\gamma^2 + 2u'\beta\gamma + 2v'\gamma\alpha + 2w'\alpha\beta = 0.$$

Let $$\frac{\alpha - \alpha'}{\lambda} = \frac{\beta - \beta'}{\mu} = \frac{\gamma - \gamma'}{\nu},$$

$$\frac{\alpha - \alpha'}{\lambda'} = \frac{\beta - \beta'}{\mu'} = \frac{\gamma - \gamma'}{\nu'},$$

be two equal diameters: the condition of their equality is, by Art. 13,

$$f(\lambda, \mu, \nu) = f(\lambda', \mu', \nu') \text{ .............. } (1).$$

The given line will be parallel to an axis, provided it make equal angles with these diameters; *i.e.*, (Art. 9) provided

$$l(\lambda + \lambda') + m(\mu + \mu') + n(\nu + \nu') = 0 \text{......... } (2).$$

This assumes $\lambda$, $\mu$, $\nu$, $\lambda'$, $\mu'$, $\nu'$ to be absolute direction-sines, therefore by equations (iii.) of Art. 3, p. 45,

$$2(\mu\nu - \mu'\nu')\cos A + \mu^2 - \mu'^2 + \nu^2 - \nu'^2 = 0,$$
$$2(\nu\lambda - \nu'\lambda')\cos B + \nu^2 - \nu'^2 + \lambda^2 - \lambda'^2 = 0,$$
$$2(\lambda\mu - \lambda'\mu')\cos C + \lambda^2 - \lambda'^2 + \mu^2 - \mu'^2 = 0.$$

Using these three equations to eliminate from (1) the terms whose arguments are $\mu\nu$, $\mu'\nu'$, $\nu\lambda$, &c., we obtain

$$(\lambda^2 - \lambda'^2)(u - v'\sec B - w'\sec C) + (\mu^2 - \mu'^2)(v - w'\sec C - u'\sec A)$$
$$+ (\nu^2 - \nu'^2)(w - u'\sec A - v'\sec B) = 0 \text{......... } (3).$$

Again $\quad a\lambda + b\mu + c\nu = 0,$

and $\quad a\lambda' + b\mu' + c\nu' = 0.$

*Adding* and solving with (2), we have

$$\frac{\lambda + \lambda'}{bn - cm} = \frac{\mu + \mu'}{cl - an} = \frac{\nu + \nu'}{am - bl} \text{ ............. } (4).$$

Also *subtracting* and multiplying the three terms by the three members of (4) respectively, we have

$$\frac{a(\lambda^2 - \lambda'^2)}{bn - cm} + \frac{b(\mu^2 - \mu'^2)}{cl - an} + \frac{c(\nu^2 - \nu'^2)}{am - bl} = 0 \text{ ...... } (5).$$

Again, by equation (iv.) of Art. 4,

$$(\lambda^2 - \lambda'^2)\sin 2A + (\mu^2 - \mu'^2)\sin 2B + (\nu^2 - \nu'^2)\sin 2C = 0 \text{... } (6).$$

Eliminating $\lambda^2 - \lambda'^2 : \mu^2 - \mu'^2 : \nu^2 - \nu'^2$ from (3), (5), and (6), we obtain the final result

$$\begin{vmatrix} \dfrac{u - v'\sec B - w'\sec C}{a}, & \dfrac{v - w'\sec C - u'\sec A}{b}, & \dfrac{w - u'\sec A - v'\sec B}{c} \\ \dfrac{1}{bn - cm}, & \dfrac{1}{cl - an}, & \dfrac{1}{am - bl} \\ \cos A, & \cos B, & \cos C \end{vmatrix} = 0,$$

which, together with the relation

$$l\bar{\alpha}+m\bar{\beta}+n\bar{\gamma}=0,$$

where $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ are the coordinates of the centre, will express the conditions necessary and sufficient that the line

$$l\alpha+m\beta+n\gamma=0$$

should be an axis of the conic.

We may notice that if the given line be the line at infinity, the former of these equations of condition reduces to an identity, while the latter assumes the form of the condition that the conic should be a parabola. Our result therefore in this case expresses that the line at infinity is an axis of every parabola, and that no conic but a parabola has an axis lying wholly at infinity; properties which we know otherwise to be true.

24. The reader will observe that the results of the last two articles—expressing the conditions that equations of the form which it has been the object of this paper to illustrate, and the ordinary linear equation respectively, should be axes of a given conic—might have been immediately deduced from each other, by the formulæ of Art. 1, p. 43. We have thought fit, however, to give both the investigations, as they exemplify two classes of cases in which great simplifications may be affected by an application of the formulæ of p. 45.

Here we will stop. Sufficient, it is hoped, has been done to shew that the equations which we have investigated may sometimes be employed more advantageously than the ordinary equation to represent a straight line. In some cases, as in the problem of Art. 14, they will give a shorter solution than the ordinary form: in others, as in Art. 15, the method is geometrically more intelligible. The reader will have observed that the work of the latter article is almost identical with that which is gone through in the ordinary method, but the interpretation is different: in our method the nature of the conic is determined by considering in how many directions an infinite radius vector can be drawn; in the other, by considering in how many points the curve meets the line at infinity. Similarly our method of finding the foci is founded on the simple property that the sum of the reciprocals of the intercepts of a focal chord is constant, and avoids the geometrically unintelligible tangents through a focus, and their characteristic property of passing through the two points in the line at infinity through which all circles pass.

Little Leigh Parsonage, *September*, 1861.

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## NOTE ON CENTRAL FORCES.

In general, the problem of central forces is solved by considering the equation connecting $u$ $\left(\text{or } \frac{1}{r}\right)$ and $\theta$, and employing the resulting integrated relation between $r$ and $\theta$ to find $\theta$ in terms of $t$ from the law of equable description of areas. If we try to express $r$ and $\theta$ separately, in terms of $t$, without first determining the form of the orbit, we are led to a host of curious results which may be easily obtained; so easily indeed, that we shall merely notice one or two of them.

From the usual equations for motion about a centre, *i.e.*,

$$\frac{d^2x}{dt^2} = -P\frac{x}{r},$$

$$\frac{d^2y}{dt^2} = -P\frac{y}{r},$$

where $P$ is the acceleration due to the central force, we get at once

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = -2\int Pdr,$$

and
$$x\frac{d^2x}{dt^2} + y\frac{d^2y}{dt^2} = -Pr.$$

Adding, we have immediately,

$$\frac{d}{dt}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right) = \tfrac{1}{2}\frac{d^2(r^2)}{dt^2} = -2\int Pdr - Pr \ldots (1).$$

This, for any assigned form of $P$ in terms of $r$, will evidently give us $r^2$ in terms of $t$.

Now there is a remarkable case in which $r^2$ can be generally expressed as a rational integral function of $t$. Suppose

$$-2\int P dr - Pr = 2C \text{..................} (2),$$

and we have

$$\frac{d^2(r^2)}{dt^2} = 2C,$$

or

$$r^2 = A + 2Bt + Ct^2 \text{..................} (3).$$

From (2) we find by differentiation

$$3P + r\frac{dP}{dr} = 0,$$

or

$$P \propto \frac{1}{r^3}.$$

Hence the case in question is that of the inverse third power. It may be worth while to find $\theta$ in terms of $t$, and to obtain, by elimination of $t$, the equations to the orbits which are possible with such a force.

We have, in all central orbits,

$$r^2\frac{d\theta}{dt} = h \text{........................} (4).$$

Hence, in the present case, by (3),

$$\frac{d\theta}{dt} = \frac{h}{A + 2Bt + Ct^2} = \frac{h}{C}\cdot\frac{1}{\left(t + \frac{B}{C}\right)^2 + \frac{AC - B^2}{C^2}} \text{...} (5).$$

Put now

$$\tau = t + \frac{B}{C},$$

and we get

$$r^2 = C\left\{\tau^2 + \frac{AC - B^2}{C^2}\right\} \text{................} (3'),$$

and

$$\frac{d\theta}{d\tau} = \frac{h}{C}\,\frac{1}{\tau^2 + \frac{AC - B^2}{C^2}} \text{................} (5').$$

There are, of course, four cases.

I. $AC=B^2$. The integral of (5′) is

$$\theta+\alpha=-\frac{h}{C}\,\frac{1}{\tau};$$

and $$r=\pm\sqrt{(C)}\,\tau.$$

Here $C$ must be *positive*. Hence

$$r=\mp\frac{h}{\sqrt{(C)}\,(\theta+\alpha)},$$

the reciprocal spiral.

II. $\dfrac{AC-B^2}{C^2}=a^2$. (3′) and (5′) give

$$\frac{aC}{h}(\theta+\alpha)=\tan^{-1}\frac{\tau}{a},$$

and therefore $$r^2=Ca^2\sec^2\frac{aC}{h}(\theta+\alpha),$$

or $$r\cos\frac{aC}{h}(\theta+\alpha)=\sqrt{\left(\frac{AC-B^2}{C}\right)}.$$

III. $\dfrac{AC-B^2}{C^2}=-a^2$. Here

$$\frac{C}{h}(\theta+\alpha)=\frac{1}{2a}\log\frac{\tau-a}{\tau+a},$$

and $$r^2=C(\tau^2-a^2),$$

whence, after a reduction or two,

$$r=\frac{2a\sqrt{(C)}}{\epsilon^{\frac{aC}{h}(\theta+\alpha)}-\epsilon^{-\frac{aC}{h}(\theta+\alpha)}}=\frac{1}{M\epsilon^{\frac{aC}{h}\theta}-N\epsilon^{-\frac{aC}{h}\theta}}.$$

IV. $C=0$,

$$\theta=\frac{h}{2B}\log\frac{r}{M},\ \text{or}\ r=M\epsilon^{\frac{2B}{h}\theta}.$$

These are, of course, the results of the integration of the usual equation between $u$ and $\theta$.

As another case, suppose in (1),

$$-2\int Pdr-Pr=mr^2+\frac{C}{2}\ \ldots\ldots\ldots\ldots\ldots(6).$$

Differentiate, multiply by $r^3$, and integrate, then

$$P = -\tfrac{1}{2}mr + \frac{n}{r^3}.$$

Hence, in the case of the direct first power, or a combination of this with the inverse third,

$$\frac{d^2(r^2)}{dt^2} = 2mr^2 + C,$$

which gives, according as $m$ is positive or negative,

$$r^2 + \frac{C}{2m} = \left\{ \begin{matrix} M\varepsilon^{\sqrt{(2m)}t} + N\varepsilon^{-\sqrt{(2m)}t} \\ M\cos\{\sqrt{(-2m)}\,t + N\} \end{matrix} \right\}.$$

By means of (4), these equations give us $\theta$ in terms of $t$, and, the latter being eliminated, we have the required orbit, which becomes the ellipse or hyperbola as usual when $n = 0$, it being observed that we have an additional disposable constant introduced by the method employed in obtaining equation (1). It is evident that results of this kind may be multiplied indefinitely. To classify the cases in which the equations for $r^2$ and $\theta$ in terms of $t$ can be completely integrated would be an interesting, but by no means an easy problem.

T.

## SHORT PROOF OF THE MULTINOMIAL THEOREM FOR A POSITIVE INTEGRAL EXPONENT.

In order to find the product of

$$(a_1 + b_1 + c_1 + \ldots + p_1)(a_2 + b_2 + c_2 + \ldots + p_2)(a_3 + b_3 + c_3 + \ldots + p_3) \\ \ldots (a_n + b_n + c_n + \ldots + p_n),$$

we have by continued multiplication to combine each member of the first group with each member of the second group, and to combine each of these combinations with each member of the third group, and so on till we arrive at the last group. Each of the final combinations will evidently be of the $n^{\text{th}}$ order, and the sum of them will be the product required.

The number of combinations containing $n$ $a$'s is *one*.

The number of combinations containing $(n-1)$ $a$'s and one $b$ is $n$. For the number of combinations that can be made by taking $(n-1)$ things at a time out of the $n$ things $a_1, a_2, a_3, \ldots a_n$ is $n$; and there is only one $b$ which can be combined with each such combination, viz., that $b$ which was not in the same group with any of the $a$'s in that particular combination.

By continuing the same argument we have generally; the number of combinations containing $r_1$ $a$'s, $r_2$ $b$'s, $r_3$ $c$'s, $\ldots r_p$ $p$'s (where $r_1+r_2+r_3+\ldots+r_p=n$) must equal the number of combinations of $r_1$ $a$'s taken out of a set of $n$ $a$'s, with $r_2$ $b$'s taken out of a set of $(n-r)$ $b$'s; with $r_3$ $c$'s taken out of a set of $\{n-(r_1+r_2)\}$ $c$'s, ... with $r_p$ $p$'s taken out of a set of $\{n-(r_1+r_2+r_3+\ldots+r_{p-1})\}$ $p$'s, *i.e.*, of $r_p$ $p$'s; this number is, of course,

$$\frac{n(n-1)\ldots(n-r_1+1)}{1.2.3\ldots r_1} \times \frac{(n-r_1)(n-r_1+1)\ldots(n-r_1-r_2+1)}{1.2.3\ldots r_2}$$

$$\times \frac{\{n-(r_1+r_2)\}\,\{n-(r_1+r_2)-1\}\ldots\{n_1-(r_1+r_2+r_3)+1\}}{1.2.3\ldots r_3}$$

$$\times \ldots\ldots \times \frac{r_p(r_p-1)\ldots 2.1}{1.2.3\ldots r_p},$$

or

$$\frac{\lfloor\underline{n}}{\lfloor\underline{r_1}, \lfloor\underline{r_2}, \lfloor\underline{r_3}, \cdots \lfloor\underline{r_p}}.$$

Now let us put

$$a_1 = a_2 = a_3 = \ldots = a_n = a,$$

$$b_1 = b_2 = b_3 = \ldots = b_n = b,$$

&c., &c.,

then each of these combinations becomes $a^{r_1}, b^{r_2}, c^{r_3}, \ldots p^{r_p}$, and

$$(a_1+b_1+c_1+\ldots+p_1)(a_2+b_2+c_2+\ldots+p_2)\ldots(a_n+b_n+c_n+\ldots+p_3$$
$$=(a+b+c+\ldots+p)^n.$$

Therefore the general term of the expansion of

$$(a+b+c+\ldots+p)^n$$

is

$$\frac{\lfloor\underline{n}}{\lfloor\underline{r_1}, \lfloor\underline{r_2}, \lfloor\underline{r_3}, \cdots \lfloor\underline{r_p}} a^{r_1}, b^{r_2}, c^{r_3}, \ldots p^{r_p}.$$

# A METHOD OF FINDING THE LATITUDE OF A PLACE BY DIRECT DIFFERENTIAL OBSERVATIONS.

By *A. Freeman, B.A.*, Scholar of St. John's College, Cambridge.

THIS method is a modification of one given in Loomis' *Practical Astronomy*. By one additional observation on each occasion of the star's crossing the prime vertical, we obviate the necessity of knowing the star's declination.

Suppose a transit circle placed so that its graduated circle is parallel to the prime vertical. Every star whose declination is less than the latitude will cross the prime vertical twice in 24 hours, at an interval (varying for different stars) of from 0 to 24 hours. By observing the star each time it crosses, we know the time between the two observations; and the arc of the prime vertical cut off by the star's diurnal circle is known from the angle through which the instrument is turned.

Thus, if $P$ (fig. 21) be the pole of the earth, $Z$ the zenith, $S$, $S'$ the points where the star crosses the prime vertical $SZS'$, and the dotted line the star's diurnal circle, then $SZS'$ and the angle $SPS'$ are known from the observations.

Join $PZ$, then $PZ$ is the co-latitude, and is at right angles to $SZS'$.

Therefore $SZ$ and the angle $SPZ$ are the halves of $SS'$ and the angle $SPS'$, and are therefore known.

Then, by Napier's rules, we have

$$\sin PZ = \tan SZ \cot SPZ,$$

which determines the co-latitude $PZ$. Also

$$\sin SZ = \sin PS \sin SPZ,$$

or

$$\sin PS = \sin SZ \div \sin SPZ,$$

which determines the stars N.P.D., $PS$.

In practice several corrections have to be applied, as, for refraction, and again, because the star crosses the wires of the instrument obliquely, but it is unnecessary to discuss these, as they are not peculiar to this method.

## THEOREM IN TRILINEAR COORDINATES.

By *W. E. Mullins, B.A.*, Assistant Master, Marlborough College, Scholar of St. John's College, Cambridge.

*To find the general equation in trilinear coordinates to a curve similar and similarly situated to a given curve.*

Let $\phi(\alpha, \beta, \gamma) = 0$ be the equation to the given curve, and $ABC$ the triangle of reference. Then there is some triangle $A'B'C'$ similar and similarly situated to $ABC$; and situated in the same relation to the new curve as $ABC$ is to the given curve.

Hence, if $\alpha'$, $\beta'$, $\gamma'$ be coordinates referred to the triangle $A'B'C'$, the equation $\phi(\alpha', \beta', \gamma') = 0$ will represent the new curve; for this equation being homogeneous is not disturbed by a variation of all the linear parameters in a constant ratio.

But if

$$\lambda\alpha + b\beta + c\gamma = 0, \quad a\alpha + \mu\beta + c\gamma = 0, \quad a\alpha + b\beta + \nu\gamma = 0$$

be the equations to the lines $BC$, $CA$, $AB$, respectively, we have

$$\alpha' = \frac{\lambda\alpha + b\beta + c\gamma}{\lambda - a}, \quad \beta' = \frac{a\alpha + \mu\beta + c\gamma}{\mu - b}, \quad \gamma' = \frac{a\alpha + b\beta + \nu\gamma}{\nu - c}.$$

Hence the equation to any curve similar and similarly situated to the curve $\phi(\alpha, \beta, \gamma) = 0$, is represented by the equation,

$$\phi\left\{\frac{\lambda\alpha + b\beta + c\gamma}{\lambda - a}, \frac{a\alpha + \mu\beta + c\gamma}{\mu - b}, \frac{a\alpha + b\beta + \nu\gamma}{\nu - c}\right\} = 0;$$

where $\lambda$, $\mu$, $\nu$ are constants, which can be determined when the other three conditions which such a curve can always satisfy are specified.

*To find the ratio of the linear dimensions of the two curves in terms of the undetermined constants.*

Let $\rho$ be the ratio of the linear dimensions of the new to those of the given curve.

Then $\rho a$, $\rho b$, $\rho c$ are the lengths of the sides of the triangle $A'B'C'$. Therefore twice its area

$$=\rho\left\{\frac{a(\lambda\alpha+b\beta+c\gamma)}{\lambda-a}+\frac{b(a\alpha+\mu\beta+c\gamma)}{\mu-b}+\frac{c(a\alpha+b\beta+\nu\gamma)}{\nu-c}\right\}$$

$$=\rho(a\alpha+b\beta+c\gamma)\left\{\frac{a}{\lambda-a}+\frac{b}{\mu-b}+\frac{c}{\nu-c}+1\right\}.$$

But twice the area of the triangle $ABC$ is $a\alpha+b\beta+c\gamma$, and the areas are in the ratio $\rho^2:1$; therefore

$$\rho\left\{\frac{a}{\lambda-a}+\frac{b}{\mu-b}+\frac{c}{\nu-c}+1\right\}=\rho^2,$$

or
$$\rho=1+\frac{a}{\lambda-a}+\frac{b}{\mu-b}+\frac{c}{\nu-c},$$

which is therefore the ratio required.

## A NEW SOLUTION OF BIQUADRATIC EQUATIONS.

The following method is given by Schlömilch in the January number of the *Zeitschrift für Math. u. Physik*; of whose paper this is merely a simplification.

Let
$$x^4+ax^3+bx^2+cx+d=0$$

be the biquadratic equation. Writing $x'-\frac{a}{4}$ for $x$ we can at once get rid of the second term, and may therefore start with the equation

$$x^4+bx^2+cx+d=0.$$

For $x$ write $py+q$, whence

$$p^4y^4+4p^3qy^3+(6p^2q^2+bp^2)y^2+(4pq^3+2bpq+cp)y$$
$$+(q^4+bq^2+cq+d)=0\text{.........}(1).$$

Let $p$ and $q$ be determined by the equations

$$p^4=q^4+bq^2+cq+d\text{..................}(2),$$
$$4p^2q=4q^3+2bq+c.$$

Eliminating $p$, we get the cubic for $q$,

$$8cq^3+(16d-4b^2)q^2-4bcq-c^2=0,$$

and from its real root, (2) gives a value for $p$, (1) then becomes a reciprocal equation, and can easily be solved.

# TRILINEAR COORDINATES.

By *W. Esson, B.A.*, Fellow and Tutor of Merton College, Oxford.

1. Let $ABC$ (fig. 22) be the triangle of reference, $A'B'C'$ any right line cutting its sides in $A'B'C'$; $AL$, $BM$, $CN$ perpendiculars from its vertices on the right line. Draw $Ol'$ (fig. 23) parallel to these perpendiculars, and from a point $O$ in $Ol'$ draw $Oa'$, $Ob'$, $Oc'$ parallel to $BC$, $CA$, $AB$, and in the same directions. Let the angular distances of $Oa'$, $Ob'$, $Oc'$ be measured from $l$ in the direction $b'c'a'$, and let $\angle la'=\theta$, $\angle lb'=\phi$, $\angle lc'=\psi$ be called the direction-angles of the right line. Since $\angle b'c'=\pi-A$, $c'a'=\pi-B$, $a'b'=\pi-C$, we have the following relations:

$$\left.\begin{aligned}\pi-A&=\psi-\phi\\ \pi-B&=\theta-\psi\\ \pi-C&=2\pi+\phi-\theta\end{aligned}\right\}\text{.................}(1);$$

therefore

$$\left.\begin{aligned}\sin A+\sin(\phi-\psi)&=0\\ \sin B+\sin(\psi-\theta)&=0\\ \sin C+\sin(\theta-\phi)&=0\end{aligned}\right\}\text{...............}(2),$$

$$\left.\begin{aligned}\cos A+\cos(\phi-\psi)&=0\\ \cos B+\cos(\psi-\theta)&=0\\ \cos C+\cos(\theta-\phi)&=0\end{aligned}\right\}\text{...............}(3),$$

which are symmetrical relations between the direction-angles of a right line and the angles of the triangle of reference.

Let $AL=p$, $BM=q$, $CN=r$, then

$$\left.\begin{aligned}\cos\theta&=\frac{q-r}{a}\\ \cos\phi&=\frac{r-p}{b}\\ \cos\psi&=\frac{p-q}{c}\end{aligned}\right\}\text{.....................}(4),$$

symmetrical relations between the direction-cosines of a right line and its direction-perpendiculars.

2. The equation of a right line in terms of $p$, $q$, $r$.

By similar triangles, (fig. 22),

$$\frac{MA'}{q} - \frac{NA'}{r} = 0,$$

$$\frac{NB'}{r} - \frac{LB'}{p} = 0,$$

$$\frac{LC'}{p} - \frac{MC'}{q} = 0.$$

Adding, $$\frac{B'C'}{p} + \frac{C'A'}{q} + \frac{A'B'}{r} = 0 \ldots\ldots\ldots\ldots\ldots (5),$$

but, if $\alpha$, $\beta$, $\gamma$ be the coordinates of any point $P$ in $A'B'C'$,

$$\alpha = PA' \cos\theta = PA' \frac{q-r}{a}, \text{ from } (4);$$

therefore $$a\alpha = (q-r)\, PA'.$$

Similarly $$b\beta = (r-p)\, PB',$$

$$c\gamma = (p-q)\, PC',$$

hence $p a\alpha + q b\beta + r c\gamma + qrB'C' + rpC'A' + pqA'B' = 0$,

but from (5), $qrB'C' + rpC'A' + pqA'B' = 0$;

therefore $$pa\alpha + qb\beta + rc\gamma = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

is the equation required.

3. The perpendicular from any point on a right line.

Draw a right line through the point parallel to the given right line. Then if $\alpha'$ be the perpendicular required, the equation of the parallel line is plainly

$$(p-\alpha')\, a\alpha + (q-\alpha')\, b\beta + (r-\alpha')\, c\gamma = 0;$$

therefore $$\alpha' = \frac{pa\alpha + qb\beta + rc\gamma}{a\alpha + b\beta + c\gamma}$$

$$= \frac{1}{2\Delta}(pa\alpha + qb\beta + rc\gamma) \ldots\ldots\ldots\ldots\ldots (7);$$

hence, if $l$, $m$, $n$ are the ratios of the direction-perpendiculars of any right line $(lmn)$ to the corresponding direction-perpendiculars of the three sides of the triangle of reference, $(lmn)$ is the perpendicular from any point on that line. In what follows, $l$, $m$, $n$ will be limited to these values.

4. From the geometry (of fig. 24), we have, regarding the *signs* of lines as well as their magnitude,

$$LA.MN+MB.NL+NC.LM$$
$$=2\,(BOC+COA+AOB)=2\Delta\,;$$

therefore $\quad pa\sin\theta+qb\sin\phi+rc\sin\psi=2\Delta,$

or $\quad l\sin\theta+m\sin\phi+n\sin\psi=1 \ldots\ldots\ldots\ldots(8),$

but from (2) and (3),

$$\cos\phi\sin B-\cos\psi\cos C=-\sin\theta\sin A,$$

so that, from (4),

$$\sin\theta=\frac{1}{bc\sin A}\,\{(p-q)\,b\cos C-(r-p)\,c\cos B\}$$
$$=\frac{1}{2\Delta}\,\{pa-qb\cos C-rc\cos B\}\,;$$

therefore, similarly,

$$\left.\begin{aligned}\sin\theta&=\,l-m\cos C-n\cos B,\\ \sin\phi&=m-n\cos A-l\cos C,\\ \sin\psi&=\,n-l\cos B-m\cos A,\end{aligned}\right\}\ldots\ldots\ldots\ldots(9).$$

We have therefore, from (8) and (9),

$$l^2+m^2+n^2-2mn\cos A-2nl\cos B-2lm\cos C=1 \ldots\ldots(10).$$

5. To reduce any other equation of the form

$$La+M\beta+N\gamma=0,\quad (LMN),$$

to the form ($lmn$).

From (10) it appears that the reduction is effected by dividing the equation ($LMN$) by

$$\{L^2+M^2+N^2-2MN\cos A-2NL\cos B-2LM\cos C\}^{\frac{1}{2}},$$

so that we need only consider the equation in the form ($lmn$).

6. Transformation of coordinates.

Let the equations of any three right lines be

$$\left.\begin{aligned}\alpha'&=l_1\alpha+m_1\beta+n_1\gamma=0\\ \beta'&=l_2\alpha+m_2\beta+n_2\gamma=0\\ \gamma'&=l_3\alpha+m_3\beta+n_3\gamma=0\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(11),$$

where $\alpha'$, $\beta'$, $\gamma'$ are the perpendiculars from any point on the three right lines. Any equation in terms of $\alpha$, $\beta$, $\gamma$ may be expressed in terms of $\alpha'$, $\beta'$, $\gamma'$ by the following substitutions:

$$\left.\begin{aligned} |\,l_1m_2n_3\,|\,\alpha &= |\,m_2n_3\,|\,\alpha' + |\,m_3n_1\,|\,\beta' + |\,m_1n_2\,|\,\gamma' \\ |\,l_1m_2n_3\,|\,\beta &= |\,n_2l_3\,|\,\alpha' + |\,n_3l_1\,|\,\beta' + |\,n_1l_2\,|\,\gamma' \\ |\,l_1m_2n_3\,|\,\gamma &= |\,l_2m_3\,|\,\alpha' + |\,l_3m_1\,|\,\beta' + |\,l_1m_2\,|\,\gamma' \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(12),$$

where $|\,l_1m_2n_3\,|$ represents the determinant

$$\begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix}, \text{ \&c.}$$

It is easily seen that the equation of $(lmn)$ referred to $(\alpha'\beta'\gamma')$ is

$$|\,lm_2n_3\,|\,\alpha' + |\,mn_3l_1\,|\,\beta' + |\,nl_1m_2\,|\,\gamma' = 0.$$

7. Values of the sides and area of the triangle $\alpha'\beta'\gamma'$ in terms of $l_1m_1n_1$, &c.

Let $(\alpha_1\beta_1\gamma_1)$, $(\alpha_2\beta_2\gamma_2)$, $(\alpha_3\beta_3\gamma_3)$ be the coordinates of the vertices of $(\alpha'\beta'\gamma')$, then we have

$$\begin{aligned} \frac{\alpha_1}{L_1} = \frac{\beta_1}{M_1} = \frac{\gamma_1}{N_1} &= \frac{l_1\alpha_1 + m_1\beta_1 + n_1\gamma_1}{|\,l_1m_2n_3\,|} = \frac{p_0'}{|\,l_1m_2n_3\,|} \\ &= \frac{a\alpha_1 + b\beta_1 + c\gamma_1}{|\,am_2n_3\,|} = \frac{2\Delta}{|\,am_2n_3\,|} \quad \ldots\ldots\ldots\ldots(13), \end{aligned}$$

where $p_0'$ is the perpendicular from $(\alpha_1\beta_1\gamma_1)$ upon $\alpha'$, and $L_1$, $M_1$, $N_1$ are the minors of the determinant $|\,l_1m_2n_3\,|$.

Symmetrical expressions may be obtained for the other vertices.

Again $a\alpha_2 + b\beta_2 + c\gamma_2 = a\alpha_3 + b\beta_3 + c\gamma_3 = 2\Delta$;

therefore $\quad |\,\gamma_2\alpha_3\,|\,c - |\,\alpha_2\beta_3\,|\,b = 2\Delta\,(\alpha_3 - \alpha_2)$,

but if $a'$ be the side joining $(\alpha_2\beta_2\gamma_2)$, $(\alpha_3\beta_3\gamma_3)$

$$\frac{\alpha_3 - \alpha_2}{a'} = \frac{q_1 - r_1}{a} = \frac{2\Delta}{abc}\,(m_1c - n_1b);$$

therefore $| \gamma_2 \alpha_3 | \, c - | \alpha_2 \beta_3 | \, b = a' . \dfrac{4\Delta^2}{abc} (m_1 c - n_1 b).$

Comparing the two forms of the equation of $a'$,

$$| \alpha \beta_2 \gamma_3 | = 0,$$

$$l_1 \alpha + m_1 \beta + n_1 \gamma = 0,$$

$$\frac{| \beta_2 \gamma_3 |}{l_1} = \frac{| \gamma_2 \alpha_3 |}{m_1} = \frac{| \alpha_2 \beta_3 |}{n_1} = \frac{| \gamma_2 \alpha_3 | \, c - | \alpha_2 \beta_3 | \, b}{m_1 c - n_1 b} = a' \, \frac{4\Delta^2}{abc}$$

$$= \frac{\alpha_1 | \beta_2 \gamma_3 | + \beta_1 | \gamma_2 \alpha_3 | + \gamma_1 | \alpha_2 \beta_3 |}{l_1 \alpha_1 + m_1 \beta_1 + n_1 \gamma_1} = \frac{| \alpha_1 \beta_2 \gamma_3 |}{p_0'} \qquad \text{.................}(14);$$

therefore $$p_0' a' = 2\Delta' = \frac{abc}{4\Delta^2} | \alpha_1 \beta_2 \gamma_3 | \qquad \text{............} (15).$$

From equations similar to (13)

$$\frac{| \beta_2 \gamma_3 |}{l_1} = \frac{| \gamma_2 \alpha_3 |}{m_1} = \frac{| \alpha_2 \beta_3 |}{n_1} = 4\Delta^2 \, \frac{| l_1 m_2 n_3 |}{| a m_3 n_1 | \, | a m_1 n_2 |} \qquad \text{..................}(16),$$

and $$\frac{\alpha_1 | \beta_2 \gamma_3 |}{l_1 L_1} = \frac{\beta_1 | \gamma_2 \alpha_3 |}{m_1 M_1} = \frac{\gamma_1 | \alpha_2 \beta_3 |}{n_1 N_1} = \frac{| \alpha_1 \beta_2 \gamma_3 |}{| l_1 m_2 n_3 |}$$

$$= (2\Delta)^3 \, \frac{| l_1 m_2 n_3 |}{| a m_2 n_3 | \, | a m_3 n_1 | \, | a m_1 n_2 |} \ \text{...}(17);$$

therefore, by comparing (14), (16); (15), (17),

$$a' = abc \, \frac{| l_1 m_2 n_3 |}{| a m_3 n_1 | \, | a m_1 n_2 |},$$

and $$\Delta' = \Delta \, \frac{abc \, | l_1 m_2 n_3 |^2}{| a m_2 n_3 | \, | b m_3 n_1 | \, | c m_1 n_2 |}.$$

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# ON LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER.

By *James Cockle, M.A.*, of Trinity College, Cambridge.

1. GIVEN $$P\frac{d^2y}{dx^2}+Q\frac{dy}{dx}+Ry=0 \qquad\text{.................(1)},$$
where $P$, $Q$, and $R$ are functions of $x$, put
$$y=e^{\int u dx}.$$
Then (1) becomes
$$e^{\int u dx}\left\{P\left(\frac{du}{dx}+u^2\right)+Qu+R\right\}=0,$$
or, rejecting the first factor, dividing by $P$ and transposing,
$$\frac{du}{dx}+\frac{Q}{P}u+u^2=-\frac{R}{P} \qquad\text{.................(2)}.$$
Let $$\frac{Q}{P}=2q,\quad \frac{R}{P}=r,$$
then (2) becomes
$$\frac{du}{dx}+2qu+u^2=-r \qquad\text{.................(3)},$$
or $$\frac{d(u+q)}{dx}+(u+q)^2=q^2+\frac{dq}{dx}-r \qquad\text{...........(4)},$$
or, making $u+q=v$, $q^2+\frac{dq}{dx}-r=s$,
$$\frac{dv}{dx}+v^2=s \qquad\text{.........................(5)}.$$

2. In the above formulæ change $u$ into $-u$. Then (3) may be put under the form
$$\frac{du}{dx}+2qu-u^2=r \qquad\text{....................(6)},$$
or $$\frac{d(u-q)}{dx}-(u-q)^2=-s \qquad\text{...............(7)},$$
which, making $v=u-q$, becomes
$$\frac{dv}{dx}-v^2=-s \qquad\text{......................(8)}.$$

3. Here it is to be observed that the change of $u$ into $-u$ converts (3) into (6), (4) into (7), and (5) into (8).

4. When a particular integral of (1) can be found, the other particular integral, and, consequently, the complete integral, can be found.

5. We have now to ascertain some of the cases in which the equations in $u$ or $v$ can be solved. Let $s$ be any constant, say $c$; then from (5) we deduce

$$\frac{1}{c-v^2}\,\frac{dv}{dx}=1,$$

whence $v$ may be determined.

6. Referring to the relation (see Art. 1),

$$s=q^2+\frac{dq}{dx}-r \text{ .................... } (9),$$

we see that if

$$q=a, \quad r=b,$$

where $a$ and $b$ are any constants, then $s$ is constant. And the differential equation

$$\frac{d^2y}{dx^2}+2a\,\frac{dy}{dx}+by=0 \text{ ................ } (10)$$

is soluble.

7. We see moreover that if

$$q=\frac{c}{x+a}, \quad r=\frac{c^2-c}{(x+a)^2}+b,$$

$c$ too being a constant, then $s$ is constant. Hence the equation

$$\frac{d^2y}{dx^2}+\frac{2c}{x+a}\,\frac{dy}{dx}+\left\{\frac{c^2-c}{(x+a)^2}+b\right\}y=0 \text{ ...... } (11)$$

is soluble.

8. Let $$s=ax^m \text{ ...................... } (12),$$

$a$ being constant. Then (5) and (8) may be dealt with as cases of Riccati's equation.

9. Let
$$2c = \frac{d\lambda}{dx} - ae^{-\lambda}, \quad r = be^{-2\lambda}.$$
Then (6) becomes
$$\frac{du}{dx} + \left(\frac{d\lambda}{dx} - ae^{-\lambda}\right)u - u^2 = be^{-2\lambda} \text{ .......... } (13),$$

which, if we substitute $e^{-\lambda}v$ for $u$ and divide the result by $e^{-\lambda}$, reduces itself to
$$\frac{dv}{dx} - ae^{-\lambda}v - e^{-\lambda}v^2 = be^{-\lambda},$$

or
$$\frac{dv}{dx} - e^{-\lambda}(av + v^2) = be^{-\lambda},$$

whence
$$\frac{dv}{dx} = e^{-\lambda}(b + av + v^2),$$

or
$$\frac{1}{b + av + v^2}\,\frac{dv}{dx} = e^{-\lambda},$$

and the determination of $v$ depends upon an integration. This soluble case was deduced *à priori* (in a form somewhat less general) from an equation corresponding to (6) by assuming
$$u = pv,$$
and discussing the conditions
$$p = e^{-\int 2c\,dx} \text{ and } r = hc^2,$$
where $h$ is constant. The first of these conditions makes the first power of $v$ disappear from the equation in $v$; the second secures a soluble form.

10. Simple results are deducible from this. Let
$$\frac{d\lambda}{dx} = \frac{n}{2}\,\frac{x^{n-1}}{x^n - 1}, \quad a = 0;$$
then
$$\lambda = \log \sqrt{(x^n - 1)}, \quad e^{-\lambda} = \frac{1}{\sqrt{(x^n - 1)}},$$
$$2c = \frac{n}{2}\,\frac{x^{n-1}}{x^n - 1}, \quad r = \frac{b}{x^n - 1},$$

and we see that

$$\frac{d^2y}{dx^2}+\frac{m}{2}\,\frac{x^{m-1}}{x^m-1}\,\frac{dy}{dx}+\frac{by}{x^m-1}=0\,\ldots\ldots\ldots(14)$$

is soluble. Let $m=2$, then (14) becomes, when cleared of fractions and after a change of signs,

$$(1-x^2)\,\frac{d^2y}{dx^2}-x\,\frac{dy}{dx}-by=0\,\ldots\ldots\ldots\ldots(15)$$

a well known soluble form.

11. To every form of

$$\frac{d^2y}{dx^2}+2q\,\frac{dy}{dx}+ry=0\,\ldots\ldots\ldots\ldots\ldots\ldots\,(16)$$

there is a cognate form. For in (16) substitute $uv$ for $y$ and divide the result by $u$. We have

$$\frac{d^2v}{dx^2}+2\left(\frac{1}{u}\,\frac{du}{dx}+q\right)\frac{dv}{dx}+\frac{1}{u}\left(\frac{d^2u}{dx^2}+2q\,\frac{du}{dx}+ru\right)=0\ldots(17).$$

Assume $$2\left(\frac{1}{u}\,\frac{du}{dx}+q\right)=2\lambda,$$

$$\frac{1}{u}\left(\frac{d^2u}{dx^2}+2q\,\frac{du}{dx}+ru\right)=\mu,$$

then $$\frac{du}{dx}+(q-\lambda)\,u=0\,\ldots\ldots\ldots\ldots\ldots\ldots\,(18),$$

$$\frac{d^2u}{dx^2}+2q\,\frac{du}{dx}+(r-\mu)\,u=0\,\ldots\ldots\ldots\ldots(19).$$

Differentiating (18), and subtracting the result from (19), we find

$$(q+\lambda)\,\frac{du}{dx}+\left(r-\mu-\frac{dq}{dx}+\frac{d\lambda}{dx}\right)u=0\,\ldots\ldots(20),$$

which must be identical with (18). Hence

$$q^2-\lambda^2=r-\mu-\frac{dq}{dx}+\frac{d\lambda}{dx}\ldots\ldots\ldots\ldots\ldots(21)$$

is the relation which connects $2\lambda$ and $\mu$ the coefficients of the transformed equation (17) with those of the original (16).

12. Simple results are deducible from this. Let

$$\lambda + q = 0,$$

then (21) becomes

$$\mu = r - 2\frac{dq}{dx},$$

and the transformed equation is

$$\frac{d^2v}{dx^2} - 2q\frac{dv}{dx} + \left(r - 2\frac{dq}{dx}\right)v = 0 \text{ ...........(22).}$$

13. When (16) is soluble the cognate form is soluble. For $v = u^{-1}y$ and $u$ is obtained from the linear differential equation (18).

14. As an example take (15), first dividing it by $(1 - x^2)$. Here

$$2\frac{dq}{dx} = -\frac{d}{dx}\left(\frac{x}{1-x^2}\right) = -\left\{\frac{1}{1-x^2} + \frac{2x^2}{(1-x^2)^2}\right\},$$

and the cognate form is

$$\frac{d^2y}{dx^2} + \frac{x}{1-x^2}\frac{dy}{dx} - \left\{\frac{1 + x^2 - b(1-x^2)}{(1-x^2)^2}\right\}y = 0 \text{ .....(23).}$$

15. In (5) let $s = -\lambda^2$, then

$$0 = \frac{dv}{dx} + v^2 + \lambda^2 = \frac{dv}{dx} + (v + \lambda)^2 - 2\lambda v$$

$$= \frac{d(v+\lambda)}{dx} + (v+\lambda)^2 - 2\lambda(v+\lambda) + 2\lambda^2 - \frac{d\lambda}{dx},$$

or, putting $v + \lambda = w$,

$$\frac{dw}{dx} - 2\lambda w + w^2 + 2\lambda^2 - \frac{d\lambda}{dx} = 0.$$

Hence, if $\quad 2\lambda^2 - \frac{d\lambda}{dx} = 0, \quad$ or $\quad \frac{1}{\lambda^2}\frac{d\lambda}{dx} = 2,$

then $\quad -\frac{1}{\lambda} = 2(x + c), \quad \lambda = -\frac{1}{2(x+c)},$

and the solution of

$$\frac{dv}{dx} + v^2 = -\frac{1}{4(x+c)^2} \text{ .................(24)}$$

may be made to depend upon that of

$$\frac{dw}{dx} - 2\lambda w + w^2 = 0,$$

or

$$\frac{dw}{dx} + \frac{w}{x+c} + w^2 = 0 \text{ ................ } (25).$$

16. So, the solution of

$$\frac{dv}{dx} - v^2 = \frac{1}{4(x+c)^2} \text{ .................. } (26)$$

may be made to depend upon that of

$$\frac{dw}{dx} - \frac{w}{x+c} - w^2 = 0 \text{ ............... } (27).$$

17. Each of the equations (25) and (27) is soluble. For, dividing either of them by $w^2$, we have

$$\frac{1}{w^2} \cdot \frac{dw}{dx} \pm \frac{1}{x+c} \cdot \frac{1}{w} \pm 1 = 0,$$

or, making $\frac{1}{w} = z$ and changing signs,

$$\frac{dz}{dx} \mp \frac{z}{x+c} \mp 1 = 0,$$

a linear equation in $z$.

18. It follows that the equation

$$\frac{d^2y}{dx^2} + 2q\frac{dy}{dx} + \left\{q^2 + \frac{dq}{dx} + \frac{1}{4(x+c)^2}\right\} y = 0 \text{ ....} (28)$$

is soluble.

19. Reverting to equation (9), let

$$q = ax, \quad r = a^2x^2 + b,$$

then

$$s = a - b$$

a constant. Hence the equation

$$\frac{d^2y}{dx^2} + 2ax\frac{dy}{dx} + (a^2x^2 + b)\, y = 0 \text{ ............} (29)$$

is soluble.

20. Let $$q = \frac{a+X}{2}, \quad r = aX,$$

$X$ being any function whatever of $x$. Then

$$s = \tfrac{1}{2}\frac{dX}{dx} + \tfrac{1}{4}(a-X)^2,$$

and (5) becomes

$$\frac{dv}{dx} + v^2 = \tfrac{1}{2}\frac{d(X-a)}{dx} + \tfrac{1}{4}(X-a)^2,$$

an equation which is satisfied by taking

$$v = \frac{X-a}{2}.$$

Hence $$u = v - q = -a,$$

$$y = e^{-\int a dx} = ce^{-ax},$$

$c$ being an arbitrary constant. And this value of $y$ satisfies

$$\frac{d^2y}{dx^2} + (a+X)\frac{dy}{dx} + aXy = 0 \text{ ............ } (30).$$

4, Pump Court, Temple, London,
*December* 24, 1861.

(*To be continued.*)

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# ON THE GENERALIZATION OF CERTAIN TRIGONOMETRICAL FORMULÆ.

By *John Blissard, M.A.*

MATHEMATICAL writers have exhibited as results of the Differential and Integral Calculus the following formulæ:

$$\frac{\cos\theta}{1} - \frac{\cos 2\theta}{2} + \frac{\cos 3\theta}{3} - \&c. = \log\left(2\cos\frac{\theta}{2}\right),$$

$$\frac{\cos\theta}{1} + \frac{\cos 2\theta}{2} + \frac{\cos 3\theta}{3} + \&c. = -\log\left(2\sin\frac{\theta}{2}\right),$$

$$\frac{\sin\theta}{1} + \frac{\sin 2\theta}{2} + \frac{\sin 3\theta}{3} - \&c. = \frac{\pi-\theta}{2},$$

$$\frac{\cos\theta}{1} - \frac{\cos 3\theta}{3} + \frac{\cos 5\theta}{5} - \&c. = \frac{\pi}{4}, \quad \&c., \&c., \&c.$$

It may be useful to shew that such formulæ may be obtained and extensively generalized by means of common algebra.

1. Required to prove that ($m$ being arbitrary and $n$ a positive integer)

$$\frac{1}{m} - \frac{n}{1}\,\frac{1}{m+1} + \frac{n(n-1)}{1.2}\,\frac{1}{m+2} - \frac{n(n-1)(n-2)}{1.2.3}\,\frac{1}{m+3} + \&c.$$
$$= \frac{1.2.3\ldots n}{m(m+1)(m+2)\ldots(m+n)}.$$

This equation holds good when $n=1$ and $n=2$. Assume therefore that it holds good for some particular value of $n$, then, by putting $n+1$ for $n$, the left-hand side becomes

$$\frac{1}{m} - \frac{n+1}{1}\,\frac{1}{m+1} + \frac{(n+1)n}{1.2}\,\frac{1}{m+2} - \&c.,$$

which $= \dfrac{1}{m} - \dfrac{n}{1}\,\dfrac{1}{m+1} + \dfrac{n(n-1)}{1.2}\,\dfrac{1}{m+2} - \&c.$

$$- \left(\frac{1}{m+1} - \frac{n}{1}\,\frac{1}{m+2} + \frac{n(n-1)}{1.2}\,\frac{1}{m+3} - \&c.\right),$$

which, by hypothesis,

$$= \frac{1.2.3\ldots n}{m(m+1)\ldots(m+n)} - \frac{1.2.3\ldots n}{(m+1)(m+2)\ldots(m+n+1)}$$
$$= \frac{1.2.3\ldots(n+1)}{m(m+1)\ldots(m+n+1)},$$

that is, the above equation, if it holds for one value of $n$, holds also for the next, and therefore for every higher value of $n$; but it does hold for $n=1$ and $n=2$, therefore it holds for $n=3$, $n=4$, &c., *i.e.* for every positive integral value of $n$.

2. Required to expand $(1+x)^n \log(1+x)$,

$$(1+x)^n \log(1+x) = \left(1 + \frac{n}{1}x + \frac{n(n-1)}{1.2}x^2 + \&c.\right)\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \&c.\right)$$
$$= x + \left(n - \frac{1}{2}\right)x^2 + \left\{\frac{n(n-1)}{1.2} - \frac{n}{1}\,\frac{1}{2} + \frac{1}{3}\right\}x^3$$
$$+ \left\{\frac{n(n-1)(n-2)}{1.2.3} - \frac{n(n-1)}{1.2}\,\frac{1}{2} + \frac{n}{1}\,\frac{1}{3} - \frac{1}{4}\right\}x^4 + \&c.$$

$$+\left\{1-\frac{n}{1}\,\frac{1}{2}+\frac{n(n-1)}{1.2}\,\frac{1}{3}-\&c.\right\}x^{n+1}$$

$$-\left\{\frac{1}{2}-\frac{n}{1}\,\frac{1}{3}+\frac{n(n-1)}{1.2}\,\frac{1}{4}-\&c.\right\}x^{n+2}-\&c.$$

$$=a_1x+a_2x^2+\ldots+a_nx^n+1.2.3\ldots n\left\{\frac{x^{n+1}}{1.2.3\ldots(n+1)}\right.$$

$$\left.-\frac{x^{n+2}}{2.3\ldots(n+2)}+\frac{x^{n+3}}{3.4\ldots(n+3)}-\&c.\ (\text{by Art. }1)\right\},$$

where $a_1=1$, $a_2=n-\frac{1}{2}$, $a_3=\frac{n(n-1)}{1.2}-\frac{n}{1}\,\frac{1}{2}+\frac{1}{3}$,

$$a_4=\frac{n(n-1)(n-2)}{1.2.3}-\frac{n(n-1)}{1.2}\,\frac{1}{2}+\frac{n}{1}\,\frac{1}{3}-\frac{1}{4},\ \&c.,$$

the law of the formation of the coefficients $a_1, a_2, a_3\ldots a_n$ being evident.

3. It being assumed that $\varepsilon^{\theta\sqrt{(-1)}}+\varepsilon^{-\theta\sqrt{(-1)}}=2\cos\theta$, let $x=\varepsilon^{\theta\sqrt{(-1)}}$, then the following results are easily obtained and will be made use of.

(1) $\log x=\theta\sqrt{(-1)}$, $\log x^{-1}=-\theta\sqrt{(-1)}$.

(2) $x=\cos\theta+\sqrt{(-1)}\sin\theta$, $x^{-1}=\cos\theta-\sqrt{(-1)}\sin\theta$.

(3) $x^n+x^{-n}=2\cos n\theta$, $x^n-x^{-n}=2\sqrt{(-1)}\sin n\theta$,

$x^n=\cos n\theta+\sqrt{(-1)}\sin n\theta$, $x^{-n}=\cos n\theta-\sqrt{(-1)}\sin n\theta$.

(4) $1+x=1+\cos\theta+\sqrt{(-1)}\sin\theta$

$$=2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2}+\sqrt{(-1)}\sin\frac{\theta}{2}\right)=2\cos\frac{\theta}{2}.x^{\frac{1}{2}},\quad 1+x^{-1}=2\cos\frac{\theta}{2}.x^{-\frac{1}{2}}.$$

(5) $1-x=2\sin\frac{\theta}{2}\left(\sin\frac{\theta}{2}-\sqrt{(-1)}\cos\frac{\theta}{2}\right)$

$$=-2\sqrt{(-1)}\sin\frac{\theta}{2}.x^{\frac{1}{2}}=2\varepsilon^{-\frac{1}{2}\pi\sqrt{(-1)}}\sin\frac{\theta}{2}.x^{\frac{1}{2}},$$

and $1-x^{-1}=2\sqrt{(-1)}\sin\frac{\theta}{2}.x^{-\frac{1}{2}}=2\varepsilon^{\frac{1}{2}\pi\sqrt{(-1)}}\sin\frac{\theta}{2}.x^{-\frac{1}{2}}$,

(since $\pm\sqrt{(-1)}=\varepsilon^{\pm\frac{1}{2}\pi\sqrt{(-1)}}$).

(6) $(1+x)^n=\left(2\cos\frac{\theta}{2}\right)^n x^{\frac{1}{2}n}$, $(1+x^{-1})^n=\left(2\cos\frac{\theta}{2}\right)^n x^{-\frac{1}{2}n}$,

$$(1-x)^n=\left(2\sin\frac{\theta}{2}\right)^n\varepsilon^{-\frac{1}{2}n\pi\sqrt{(-1)}}x^{\frac{1}{2}n},\quad(1-x^{-1})^n=\left(2\sin\frac{\theta}{2}\right)^n\varepsilon^{\frac{1}{2}n\pi\sqrt{(-1)}}x^{-\frac{1}{2}n}.$$

$$(7)\quad \varepsilon^{-\frac{1}{2}n\pi\sqrt{(-1)}}\,x^{m}+\varepsilon^{\frac{1}{2}n\pi\sqrt{(-1)}}\,x^{-m}=2\cos\left(\frac{n\pi}{2}-m\theta\right),$$

$$\varepsilon^{-\frac{1}{2}n\pi\sqrt{(-1)}}\,x^{m}-\varepsilon^{\frac{1}{2}n\pi\sqrt{(-1)}}\,x^{-m}=-2\sqrt{(-1)}\sin\left(\frac{n\pi}{2}-m\theta\right).$$

By use of the above results, whatever form $f(x)$ may assume, it is evident that $f(x)\pm f(x^{-1})$ can be expressed in terms of trigonometrical functions of $\theta$.

4. Assume $f(x)=x^{m}(1+x)^{n}\log(1+x)$

which (Art. 3) $=\left(2\cos\frac{\theta}{2}\right)^{n} x^{m+\frac{1}{2}n}\log\left(2\cos\frac{\theta}{2}\,x^{\frac{1}{2}}\right)$;

therefore $f(x^{-1})=\left(2\cos\frac{\theta}{2}\right)^{n} x^{-(m+\frac{1}{2}n)}\log\left(2\cos\frac{\theta}{2}\,x^{-\frac{1}{2}}\right)$;

therefore
$$f(x)+f(x^{-1})=\left(2\cos\frac{\theta}{2}\right)^{n}\left(x^{m+\frac{1}{2}n}+x^{-(m+\frac{1}{2}n)}\right)\log 2\cos\frac{\theta}{2}$$
$$+\left(2\cos\frac{\theta}{2}\right)^{n}\left(x^{m+\frac{1}{2}n}-x^{-(m+\frac{1}{2}n)}\right)\frac{\theta\sqrt{(-1)}}{2}$$
$$=\left(2\cos\frac{\theta}{2}\right)^{n}\left\{\log\left(2\cos\frac{\theta}{2}\right).2\cos\left(m+\frac{n}{2}\right)\theta-\theta\sin\left(m+\frac{n}{2}\right)\theta\right\}.$$

Also, by the expansion of Art. 2,

$$f(x)+f(x^{-1})=a_1\left(x^{m+1}+x^{-(m+1)}\right)+a_2\left(x^{m+2}+x^{-(m+2)}\right)+\ldots$$
$$+a_n\left(x^{m+n}+x^{-(m+n)}\right)$$
$$+1.2.3\ldots n\left\{\frac{x^{m+n+1}+x^{-(m+n+1)}}{1.2.3\ldots(n+1)}-\frac{x^{m+n+2}+x^{-(m+n+2)}}{2.3.4\ldots(n+2)}\right.$$
$$\left.+\frac{x^{m+n+3}+x^{-(m+n+3)}}{3.4\ldots(n+3)}-\&c.\right\};$$

therefore
$$f(x)+f(x^{-1})=2\left\{a_1\cos(m+1)\theta+a_2\cos(m+2)\theta+\ldots\right.$$
$$\left.+a_n\cos(m+n)\theta\right\}$$
$$+2.1.2.3\ldots n\left\{\frac{\cos(m+n+1)\theta}{1.2.3\ldots(n+1)}-\frac{\cos(m+n+2)\theta}{2.3.4\ldots(n+2)}\right.$$
$$\left.+\frac{\cos(m+n+3)\theta}{3.4\ldots(n+3)}-\&c.\right\}.$$

Hence by equating the two values of $f(x)+f(x^{-1})$, we have, on reduction,

$$\frac{\cos(m+n+1)\,\theta}{1.2.3\ldots(n+1)}-\frac{\cos(m+n+2)\,\theta}{2.3.4\ldots(n+2)}+\frac{\cos(m+n+3)\,\theta}{3.4.5\ldots(n+3)}-\&\text{c.}$$

$$=\frac{\left(2\cos\frac{\theta}{2}\right)^n}{1.2.3\ldots n}\left\{\cos\left(m+\frac{n}{2}\right)\theta\,\log\left(2\cos\frac{\theta}{2}\right)-\frac{\theta}{2}\sin\left(m+\frac{n}{2}\right)\theta\right\}$$

$$-\frac{1}{1.2.3\ldots n}\{a_1\cos(m+1)\,\theta+a_2\cos(m+2)\,\theta+\ldots+a_n\cos(m+n)\,\theta\}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\text{(I)}.$$

Similarly it may be shewn that by equating the two values of $f(x)-f(x^{-1})$, we obtain

$$\frac{\sin(m+n+1)\,\theta}{1.2.3\ldots(n+1)}-\frac{\sin(m+n+2)\,\theta}{2.3.4\ldots(n+2)}+\frac{\sin(m+n+3)\,\theta}{3.4.5\ldots(n+3)}-\&\text{c.}$$

$$=\frac{\left(2\cos\frac{\theta}{2}\right)^n}{1.2.3\ldots n}\left\{\sin\left(m+\frac{n}{2}\right)\theta\,\log\left(2\cos\frac{\theta}{2}\right)+\frac{\theta}{2}\cos\left(m+\frac{n}{2}\right)\theta\right\}$$

$$-\frac{1}{1.2.3\ldots n}\{a_1\sin(m+1)\,\theta+a_2\sin(m+2)\,\theta+\ldots+a_n\sin(m+n)\,\theta\}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\text{(II)}.$$

5. Assume $f(x)=x^m(1-x)^n\log(1-x)$ which (by Art. 3)

$$=\left(2\sin\frac{\theta}{2}\right)^n x^{m+\frac{1}{2}n}\,\varepsilon^{-\frac{1}{2}n\pi\sqrt{(-1)}}\log\left(2\varepsilon^{-\frac{1}{2}\pi\sqrt{(-1)}}.\sin\frac{\theta}{2}.x^{-\frac{1}{2}}\right)$$

$$=\left(2\sin\frac{\theta}{2}\right)^n x^{m+\frac{1}{2}n}\,\varepsilon^{-\frac{1}{2}n\pi\sqrt{(-1)}}\left\{\log\left(2\sin\frac{\theta}{2}\right)-\frac{\pi-\theta}{2}\sqrt{(-1)}\right\}.$$

Similarly

$$f(x^{-1})=\left(2\sin\frac{\theta}{2}\right)^n x^{-(m+\frac{1}{2}n)}\,\varepsilon^{\frac{1}{2}n\pi\sqrt{(-1)}}\left\{\log\left(2\sin\frac{\theta}{2}\right)+\frac{\pi-\theta}{2}\sqrt{(-1)}\right\};$$

therefore

$$f(x)+f(x^{-1})=\left(2\sin\frac{\theta}{2}\right)^n\left\{\log\left(2\sin\frac{\theta}{2}\right)(\varepsilon^{-\frac{1}{2}n\pi\sqrt{(-1)}}x^{m+\frac{1}{2}n}+\varepsilon^{\frac{1}{2}n\pi\sqrt{(-1)}}x^{-(m+\frac{1}{2}n)})\right.$$

$$\left.-\frac{\pi-\theta}{2}\sqrt{(-1)}\,(\varepsilon^{-\frac{1}{2}n\pi\sqrt{(-1)}}x^{m+\frac{1}{2}n}-\varepsilon^{\frac{1}{2}n\pi\sqrt{(-1)}}x^{-(m+\frac{1}{2}n)}\right\},$$

$$\text{(by Art. 3)} = \left(2\sin\frac{\theta}{2}\right)^n \left[\log\left(2\sin\frac{\theta}{2}\right) 2\cos\left\{\frac{n\pi}{2} - \left(m+\frac{n}{2}\right)\theta\right\} - \frac{\pi-\theta}{2}\, 2\sin\left\{\frac{n\pi}{2} - \left(m+\frac{n}{2}\right)\theta\right\}\right],$$

but $f(x)+f(x^{-1})$ (by expansion in Art. 2)

$$= -2\left\{a_1\cos(m+1)\theta - a_2\cos(m+2)\theta + \ldots + (-1)^{n+1}a_n\cos(m+n)\theta\right\}$$
$$+ (-1)^{n+1}(1.2.3\ldots n)\, 2\left\{\frac{\cos(m+n+1)\theta}{1.2.3\ldots(n+1)} + \frac{\cos(m+n+2)\theta}{2.3.4\ldots(n+2)} + \frac{\cos(m+n+3)\theta}{3.4\ldots(n+3)} + \&c.\right\};$$

therefore

$$(-1)^{n+1}\left\{\frac{\cos(m+n+1)\theta}{1.2.3\ldots(n+1)} + \frac{\cos(m+n+2)\theta}{2.3.4\ldots(n+2)} + \frac{\cos(m+n+3)\theta}{3.4\ldots(n+3)} + \&c.\right\}$$
$$= \frac{\left(2\sin\frac{\theta}{2}\right)^n}{1.2.3\ldots n}\left[\log\left(2\sin\frac{\theta}{2}\right)\cos\left\{\frac{n\pi}{2} - \left(m+\frac{n}{2}\right)\theta\right\} - \frac{\pi-\theta}{2}\sin\left\{\frac{n\pi}{2} - \left(m+\frac{n}{2}\right)\theta\right\}\right]$$
$$+ \frac{1}{1.2.3\ldots n}\left\{a_1\cos(m+1)\theta - a_2\cos(m+2)\theta + \ldots + (-1)^{n+1}a_n\cos(m+n)\theta\right\}\ldots\ldots\ldots\text{(III)}.$$

Similarly, by equating the two values of $f(x)-f(x^{-1})$, we obtain

$$(-1)^n\left\{\frac{\sin(m+n+1)\theta}{1.2.3\ldots(n+1)} + \frac{\sin(m+n+2)\theta}{2.3.4\ldots(n+2)} + \frac{\sin(m+n+3)\theta}{3.4.5\ldots(n+3)} + \&c.\right\}$$
$$= \frac{\left(2\sin\frac{\theta}{2}\right)}{1.2.3\ldots n}\left[\log\left(2\sin\frac{\theta}{2}\right)\sin\left\{\frac{n\pi}{2} - \left(m+\frac{n}{2}\right)\theta\right\} + \frac{\pi-\theta}{2}\cos\left\{\frac{n\pi}{2} - \left(m+\frac{n}{2}\right)\theta\right\}\right]$$
$$- \frac{1}{1.2\ldots n}\left\{a_1\sin(m+1)\theta - a_2\sin(m+2)\theta + \ldots + (-1)^{n+1}a_n\sin(m+n)\theta\right\}\ldots\ldots\ldots\ldots\text{(IV)}.$$

6. In the above formulæ I., II., III., IV., $m$ is perfectly arbitrary, but $n$ restricted to zero or a positive integer. If $m=-n$, we have (from I.)

$$\frac{\cos\theta}{1.2\ldots(n+1)} - \frac{\cos 2\theta}{2.3\ldots(n+2)} + \frac{\cos 3\theta}{3.4\ldots(n+3)} - \&c.$$

$$= \frac{\left(2\cos\frac{\theta}{2}\right)^n}{1.2\ldots n}\left\{\cos\frac{n\theta}{2}\log\left(2\cos\frac{\theta}{2}\right) + \frac{\theta}{2}\sin\frac{n\theta}{2}\right\}$$

$$-\frac{1}{1.2\ldots n}\{a_1\cos(n-1)\theta + a_2\cos(n-2)\theta + \ldots + a_n\},$$

and from (II.)

$$\frac{\sin\theta}{1.2\ldots(n+1)} - \frac{\sin 2\theta}{2.3\ldots(n+2)} + \frac{\sin 3\theta}{3.4\ldots(n+3)} - \&c.$$

$$= \frac{\left(2\cos\frac{\theta}{2}\right)^n}{1.2\ldots n}\left\{\frac{\theta}{2}\cos\frac{n\theta}{2} - \sin\frac{n\theta}{2}\log\left(2\cos\frac{\theta}{2}\right)\right\}$$

$$+\frac{1}{1.2\ldots n}\{a_1\sin(n-1)\theta + a_2\sin(n-2)\theta, \&c.\}.$$

If in I., II., III., IV., $m=n=0$, we obtain

$$\frac{\cos\theta}{1} - \frac{\cos 2\theta}{2} + \frac{\cos 3\theta}{3} - \frac{\cos 4\theta}{4} + \&c. = \log\left(2\cos\frac{\theta}{2}\right) \ldots (1),$$

$$\frac{\sin\theta}{1} - \frac{\sin 2\theta}{2} + \frac{\sin 3\theta}{3} - \frac{\sin 4\theta}{4}, \&c. = \frac{\theta}{2} \ldots\ldots\ldots\ldots\ldots (2),$$

$$\frac{\cos\theta}{1} + \frac{\cos 2\theta}{2} + \frac{\cos 3\theta}{3} + \frac{\cos 4\theta}{4}, \&c. = -\log\left(2\sin\frac{\theta}{2}\right) \ldots (3),$$

$$\frac{\sin\theta}{1} + \frac{\sin 2\theta}{2} + \frac{\sin 3\theta}{3} + \frac{\sin 4\theta}{4}, \&c. = \frac{\pi-\theta}{2} \ldots\ldots\ldots\ldots (4).$$

By giving to $\theta$ different values within the proper limits, various important summations of series will be effected. Those limits are in I. and II. from $\theta=-\pi$ to $\theta=+\pi$, and in III. and IV. from $\theta=-2\pi$ to $\theta=+2\pi$.

Ex. 1. (In I. let $m=-n$, $\theta=0$),

$$\frac{1}{1.2\ldots(n+1)} - \frac{1}{2.3\ldots(n+2)} + \&c. = \frac{1}{1.2\ldots n}\{2^n\log 2 - (a_1+a_2+\ldots+a_n)\}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

Ex. 2. $\left(\text{In II. let } m = -n,\ \theta = \frac{\pi}{2}\right)$,

$$\frac{1}{1.2\ldots(n+1)} - \frac{1}{3.4\ldots(n+3)} + \&\text{c.}$$

$$= \frac{2^{\frac{1}{2}n}}{1.2\ldots n}\left\{\frac{\pi}{4}\cos\frac{n\pi}{4} - \sin\frac{n\pi}{4}\log\left(2\cos\frac{\theta}{2}\right)\right\}$$

$$+ \frac{1}{1.2\ldots n}\left\{a_1 \sin(n-1)\frac{\pi}{2} + a_2 \sin(n-2)\frac{\pi}{2} + \&\text{c.}\right\} \ldots(6).$$

7. By assigning different forms to $f(x)$, we can obtain by the above method numerous trigonometrical formulæ, some of which appear to be remarkable. I subjoin some examples which the intelligent young student, for whom this paper is chiefly intended, may work out for himself without much difficulty.

Ex. 1. Let $f(x) = \tan^{-1}x$, then from $f(x) + f(x^{-1})$ we obtain

$$\frac{\cos\theta}{1} - \frac{\cos 3\theta}{3} + \frac{\cos 5\theta}{5} - \frac{\cos 7\theta}{7} + \&\text{c.} = \frac{\pi}{4} \ldots\ldots (7).$$

Hence putting $\frac{\pi}{2} - \theta$ for $\theta$

$$\frac{\sin\theta}{1} + \frac{\sin 3\theta}{3} + \frac{\sin 5\theta}{5} + \frac{\sin 7\theta}{7} + \&\text{c.} = \frac{\pi}{4} \ldots\ldots (8),$$

from $f(x) - f(x^{-1})$ may be obtained

$$\frac{\sin\theta}{1} - \frac{\sin 3\theta}{3} + \frac{\sin 5\theta}{5} - \frac{\sin 7\theta}{7} + \&\text{c.} = \frac{1}{4}\log\left(\frac{1+\sin\theta}{1-\sin\theta}\right) \ldots(9).$$

Hence

$$\frac{\cos\theta}{1} + \frac{\cos 3\theta}{3} + \frac{\cos 5\theta}{5} + \frac{\cos 7\theta}{7} + \&\text{c.} = \frac{1}{2}\log\left(\cot\frac{\theta}{2}\right) \ldots (10).$$

Ex. 2. Let $f(x) = \tan^{-1}\frac{1}{x^m(1+x)^n}$, then from $f(x) + f(x^{-1})$ may be obtained by putting $2\theta$ for $\theta$, and $m$ for $m + \frac{n}{2}$,

$$\frac{\cos m\theta}{(2\cos\theta)^n} - \tfrac{1}{3}\frac{\cos 3m\theta}{(2\cos\theta)^{3n}} + \tfrac{1}{5}\frac{\cos 5m\theta}{(2\cos\theta)^{5n}} - \&\text{c.}$$

$$= \tfrac{1}{2}\tan^{-1}\left\{\frac{2\cos m\theta\,(2\cos\theta)^n}{(2\cos\theta)^{2n} - 1}\right\} \ldots\ldots\ldots (11),$$

and from $f(x)-f(x^{-1})$ may be got

$$\frac{\sin m\theta}{(2\cos\theta)^n}-\frac{1}{3}\,\frac{\sin 3m\theta}{(2\cos\theta)^{3n}}+\&c.$$

$$=\frac{1}{4}\log\left\{\frac{1+2\sin m\theta\,(2\cos\theta)^n+(2\cos\theta)^{2n}}{1-2\sin m\theta\,(2\cos\theta)^n+(2\cos\theta)^{2n}}\right\}\ldots\ldots(12).$$

Ex. 3. Let $f(x)=\log(1+x+x^2)$, then from $f(x)+f(x^{-1})$ putting $2\theta$ for $\theta$,

$$\frac{\cos 3\theta}{2\cos\theta}-\frac{1}{2}\,\frac{\cos 6\theta}{(2\cos\theta)^2}+\frac{1}{3}\,\frac{\cos 9\theta}{(2\cos\theta)^3}-\&c.=\log\left(\frac{1+2\cos 2\theta}{2\cos\theta}\right)$$
$$\ldots\ldots\ldots\ldots(13),$$

and $\cos 3\theta\,(2\cos\theta)-\frac{1}{2}\cos 6\theta\,(2\cos\theta)^2+\frac{1}{3}\cos 9\theta\,(2\cos\theta)^3-\&c.$

$$=\log(1+2\cos 2\theta)\ldots\ldots\ldots(14).$$

Similarly from $f(x)-f(x^{-1})$, we get

$$\theta=\frac{\sin 3\theta}{2\cos\theta}-\frac{1}{2}\,\frac{\sin 6\theta}{(2\cos\theta)^2}+\&c.\ \ldots\ldots\ldots\ldots(15),$$

$$2\theta=\sin 3\theta\,(2\cos\theta)-\frac{1}{2}\sin 6\theta\,(2\cos\theta)^2+\&c.\ldots\ldots(16).$$

Ex. 4. Let $f(x)=1+x+x^2+\ldots+x^{n+r}$

$$=1+x+\ldots+x^{r-1}+x^r(1+x+\ldots+x^n),$$

then $\cos(n+r)\,\theta\left(\dfrac{\sin n\theta}{\sin r\theta}\right)-\frac{1}{2}\cos(n+r)\,2\theta\left(\dfrac{\sin n\theta}{\sin r\theta}\right)$

$$+\frac{1}{3}\cos(n+r)\,3\theta\left(\frac{\sin n\theta}{\sin r\theta}\right)^3-\&c.=\log\left\{\frac{\sin(n+r)\,\theta}{\sin r\theta}\right\}\ \ldots.(17),$$

and $\sin(n+r)\,\theta\left(\dfrac{\sin n\theta}{\sin r\theta}\right)-\frac{1}{2}\sin(n+r)\,2\theta\left(\dfrac{\sin n\theta}{\sin r\theta}\right)^2$

$$+\frac{1}{3}\sin(n+r)\,3\theta\left(\frac{\sin n\theta}{\sin r\theta}\right)^3-\&c.=n\theta\ldots\ldots(18).$$

Ex. 5. Let $f(x)=\cos x$ and $f(x)=\sin x$ be used, then the following elegant formulæ may be obtained:

$$\tan(\cos\theta)=\frac{\dfrac{\cos\theta}{1}-\dfrac{\cos 3\theta}{1.2.3}+\dfrac{\cos 5\theta}{1.2.3.4.5}-\&c.}{1-\dfrac{\cos 2\theta}{1.2}+\dfrac{\cos 4\theta}{1.2.3.4}-\&c.}\ \ldots(19),$$

and also
$$=\frac{\frac{\sin 2\theta}{1.2}-\frac{\sin 4\theta}{1.2.3.4}+\frac{\sin 6\theta}{1.2\ldots 6}-\&c.}{\frac{\sin\theta}{1}-\frac{\sin 3\theta}{1.2.3}+\frac{\sin 5\theta}{1.2\ldots 5}-\&c.}\ldots\ldots\ldots(20).$$

These equations hold quite generally.

(1) Let $\theta=\pi$; therefore from (19),

$$\tan(-1)=\frac{-1+\frac{1}{1.2.3}-\frac{1}{1.2.3.4.5}+\&c.}{1-\frac{1}{1.2}+\frac{1}{1.2.3.4}-\&c.}=-\tan(1),$$

from (20),
$$\tan(-1)=\frac{\frac{\sin 2\pi}{1.2}-\frac{\sin 4\pi}{1.2.3.4},\ \&c.}{\frac{\sin\pi}{1}-\frac{\sin 3\pi}{1.2.3},\ \&c.}.$$

But
$$\frac{\sin n\pi}{\sin\pi}=(-1)^{n+1}n \text{ ($n$ being integral)};$$

therefore
$$\tan(-1)=\frac{-\frac{2}{1.2}+\frac{4}{1.2.3.4},\ \&c.}{\frac{1}{1}-\frac{3}{1.2.3},\ \&c.}=-\tan(1).$$

(2) Let $\theta=\frac{1}{3}\pi$, then from (20),

$$\tan\tfrac{1}{2}=\frac{\frac{1}{1.2}+\frac{1}{1.2.3.4}-\frac{1}{1.2\ldots 8}-\frac{1}{1.2\ldots 10}+\&c.}{\frac{1}{1}-\frac{1}{1.2\ldots 5}-\frac{1}{1.2\ldots 7}+\frac{1}{1.2\ldots 11}+\&c.}$$

$$=\frac{\frac{1}{1.2}+\frac{1}{1.2.3.4}}{\frac{1}{1}-\frac{1}{1.2.3.4.5}}\text{ nearly}=\frac{65}{119}\text{ nearly.}$$

Again from (19),

$$\tan(\tfrac{1}{2})=\frac{\frac{1}{2}+\frac{1}{1.2.3}+\frac{1}{2}\,\frac{1}{1.2\ldots 5}-\&c.}{1+\frac{1}{1.2}\,\frac{1}{2}-\frac{1}{2}\,\frac{1}{1.2.3.4}-\frac{1}{1.2\ldots 6}-\&c.}=\frac{483}{884}\text{ nearly.}$$

These two values are nearly identical, the difference being $\frac{483}{884} - \frac{65}{119} = \frac{1}{6188}$.

8. The preceding formulæ exhibit the summations of various infinite trigonometrical series, which, being periodic, possess only a limited range of application. It is manifestly important therefore to assign the limits within which such summations hold good.

The following rule of limits, which I suppose to be new, and which is applicable to a large class of formulæ, I have found in every instance to be correct. The proof of it will be subsequently given.

**Rule.** If such an infinite trigonometrical series as

$$\frac{\cos^m(p\theta)}{p^n} \pm \frac{\cos^m(p+r)\,\theta}{(p+r)^n} + \frac{\cos^m(p+2r)\,\theta}{(p+2r)^n} \pm \frac{\cos^m(p+3r\theta)}{(p+3r)^n} + \&c.$$

is capable of being summed in terms of the arc $\theta$, then

1st. If all the terms of the series are positive, the range of application in the summation of that series is from

$$\theta = -\frac{2\pi}{mr} \text{ to } \theta = +\frac{2\pi}{mr}.$$

2nd. If the terms are alternately positive and negative, the range of application is from

$$\theta = -\frac{\pi}{mr} \text{ to } \theta = +\frac{\pi}{mr}.$$

It may be observed that the limits themselves are *ordinarily* (I am not sure that they are universally) included within the range of application.

9. The following examples will illustrate the above rule. By the method used in my *Theory of Generic Equations*[*] the following formulæ are readily obtained.

$$(1)\quad \frac{\sin^2\theta}{1^4} - \frac{\sin^2 2\theta}{2^4} + \frac{\sin^2 3\theta}{3^4} - \&c. = \frac{\pi^2\theta^2}{12} - \frac{\theta^4}{6}.$$

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[*] See *Quarterly Journal of Mathematics*, Nos. 15 to 19.

Here the range for $\theta$ is from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$. Let $\theta = \frac{\pi}{3}$, which is within the range, then we have from (1), after reduction

$$\frac{1}{1^4} - \frac{1}{2^4} - \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{7^4} - \&c. = \frac{7\pi^4}{729},$$

which is a correct result. Let $\theta = \frac{\pi}{2}$, which is the extreme limit, we have

$$\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} \&c. = \frac{\pi^4}{96} \text{ (correct).}$$

Let $\theta = \pi$, which is beyond the range, we shall have

$$\frac{\sin^2 \pi}{1^4} - \frac{\sin^2 2\pi}{2^4}, \&c. = -\frac{\pi^2}{12},$$

*i.e.*, $\sin^2 \pi \left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2}, \&c.\right)$ or $0 = -\frac{\pi^2}{12}$ (erroneous).

Similarly, $\theta = \pm \frac{\pi}{\sqrt{(2)}}$ is beyond the range, and if substituted in (1) will give a finite quantity $= 0$.

$$(2) \quad \frac{\cos \theta}{1^4} - \frac{\cos 2\theta}{2^4} + \frac{\cos 3\theta}{3^4} - \&c. = \frac{7\pi^2}{720} - \frac{\pi^2 \theta^2}{24} + \frac{\theta^4}{48},$$

the range for $\theta$ is from $-\pi$ to $+\pi$. If $\theta = \frac{2\pi}{3}$ (within the range) we obtain a correct result. If $\theta = \pi$ (the limiting value) we also obtain a correct result. If $\theta = 2\pi$ (beyond the range) the result is erroneous. If

$$\frac{7\pi^4}{720} - \frac{\pi^2\theta^2}{24} + \frac{\theta^4}{48} = 0, \text{ and therefore } \theta = \pm \pi \sqrt{\{1 \pm 2 \sqrt{(\tfrac{2}{15})}\}},$$

of these values, $\theta = \pm \pi \sqrt{\{1 + 2 \sqrt{(\tfrac{2}{15})}\}}$ being beyond the range, will yield an erroneous, and

$$\theta = \pm \pi \sqrt{\{1 - 2 \sqrt{(\tfrac{2}{15})}\}},$$

being within the range, a correct result.

$$(3) \quad \frac{\sin^3 \theta}{1^4} - \frac{\sin^3 3\theta}{3^4} + \frac{\sin^3 5\theta}{5^4} - \&c. = \frac{\pi\theta^3}{4},$$

the range for $\theta$ is from $-\frac{\pi}{6}$ to $+\frac{\pi}{6}$. $\theta=\frac{\pi}{6}$ (the limiting value) gives a correct result, $\theta=\frac{\pi}{4}$ (beyond the range) gives

$$\frac{1}{1^4}-\frac{1}{3^4}-\frac{1}{5^4}+\frac{1}{7^4}+\frac{1}{9^4}-\&c.=\frac{\pi^4\sqrt{(2)}}{128},$$

which is erroneous.

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## SHADOW SHIFTING.

(In reply to the Query, *suprà*, p. 24).

We propose the following explanation of the phenomenon referred to. It seems that the apparent shadow of an object is not, as G. G. supposes, the geometrical shadow due to the approximate centre of the uncovered part of the sun's disc, but rather to its extreme edge. The visible shadow is the *umbra*, the penumbra being altogether disregarded, and therefore only commences where the whole of the direct sunlight is obstructed. This may be easily verified by observation, or perhaps the result may be anticipated by considering an analogous phenomenon noticed at an eclipse of the sun, viz. that so long as any the smallest part of the disc remains unobscured by the moon, the light is very inconsiderably diminished.

With this correction G. G.'s explanation will account for a displacement twice as great as without it, viz. an angular displacement equal to the whole angular diameter of the sun, varying from 31′ ½ in July to 32′ ½ in January; and this is, according to our repeated observations, the real amount of the displacement, though G. G. has estimated it "roughly" at about 40 per cent. more.

A reference to figures 25 and 26 will make the explanation clearer. *AB* represents a section of the sun's disc, *O* an object whose edge casts a shadow on the screen *ab*, *a* being the edge of the umbra and *b* of the penumbra (fig. 25). In fig. 26 the obstacle or cloud *C* is introduced, throwing its own shadow *C* on the screen and intercepting the light of the part *AP* of the sun's disc from *O*, thus shifting the edge of the shadow from *a* to *p*. If the obstacle *C* be gradually moved in the direction of the arrow it is plain that *p* will move along the screen till it come to *b*, which is the shifting for which we had to account. It will also be seen from the figure that if the obstacle *C* be dense enough to throw a distinct shadow *c* on the screen, the appearance presented will be that as the shadow *c* approaches the shadow *a*, the latter moves forward to meet it, till they do meet in *b*.

The angle which *ab* subtends at *O* is evidently the same as that which *AB* subtends, that is, it is equal to the angular diameter of the sun's disc.

It is of course not necessary that the obstacle *C* should be a cloud, nor that it should be at any great distance from *O*. We made some experiments in which the obstacle was a perfectly opaque substance, and in different experiments we varied the distance *aO* from 20 inches to 15 feet, and *OC* from 3 inches to 13 feet, but the displacement when measured and reduced was always found nearly to coincide with that given by theory.

H. & W.

*Dec.* 24, 1861.

## ANOTHER REPLY.

All the phenomena observed will be at once accounted for, when it is recollected that the edge of a cloud is not a dark opaque body, but capable of becoming highly luminous.

J. M. W.

# ADDITIONAL NOTE ON THE EQUIANGULAR SPIRAL.

(*Vide suprà*, p. 5.)

By *William Allen Whitworth, B.A.*

The reader has perhaps observed that the property enunciated in the second proposition of our former paper is not a peculiar property of the equiangular spiral, but is true for any curve whatever. However, since this spiral has *no linear parameter* though it has an *angular* one, all similar equiangular spirals are equal and capable of being superposed so as to coincide entirely. We should not, therefore, have exceeded the truth if we had said 'the locus of $Q$ will be a similar *and equal* spiral.'

An interesting extension of this proposition is that given below, which although almost as old as the spiral itself has not yet lost its interest, as is clear from its having been set as a rider in the Mathematical Tripos Examination, Monday Morning, Jan. 13, 1862.

Prop. *If rays of light diverging from the pole of an equiangular spiral be reflected and refracted at the arc, then both the caustics will be arcs of equiangular spirals similar to the original spiral.*

Let $OP$, $Op$ (fig. 27) be any two polar radii of the spiral; $OQ$, $Oq$ two other polar radii, making equal small angles $POQ$, $pOq$ with the former radii.

Let the reflected or refracted rays at $P$ and $Q$ meet in $R$ at those at $p$ and $q$ in $r$.

Then $OQPR$, $Oqpr$ are similar figures, and if $Q$, $q$ approach $P$, $p$, the angles $POQ$, $pOq$ always remaining equal to one another, $OPR$, $Opr$ will always be similar triangles. And therefore, if $Vv$ be the ultimate positions of $Rr$ when the angles $POQ$, $pOq$ ultimately diminish, $OPV$, $Opv$ will be similar triangles. But $V$, $v$ are the points on the caustic corresponding to $P$, $p$; hence, by Prop. II., the locus $Vv$ (*i.e.* the caustic) is an equiangular spiral similar to the given spiral; therefore, &c. Q.E.D.

The statement that the equiangular spiral has no linear parameter seems at first sight to be at variance with the fact that a linear constant is involved in the polar equation to the spiral. But this apparent contradiction vanishes when it is considered that the equations $r = a\varepsilon^{\mu\theta}$, $r = b\varepsilon^{\mu\theta}$, represent the same curve in two different positions, (one being turned through an angle $\frac{1}{\mu}\log\frac{a}{b}$ from the position of the other,) or in the same position, and absolutely coincident provided $a = b\varepsilon^{2r\mu\pi}$ where $r$ is any integer. The constant therefore in the equation to the spiral does not represent the magnitude of the curve, and therefore is not a parameter: it only defines the position of the curve by representing the length of that polar radius which is to coincide with the initial line.

A difficulty has been suggested with respect to our definition of the spiral given on page 5. It has been said that a circle is a limiting case of the equiangular spiral, and that its centre is the pole, and that our definition will not apply to this case.

We would ask those readers who feel this difficulty whether it may not be more correctly said that the limit of the equiangular spiral when the angle indefinitely approaches a right angle is a system of concentric circles, having their common centre at the pole, infinite in number and of every radius, so as in fact completely to cover space of two dimensions. At all events it cannot be said that a circle of any particular radius rather than of any other is the limit to which an equiangular spiral approaches when its angle approaches the value $\frac{1}{2}\pi$.

Portarlington,
*Feb.* 18, 1862.

# QUATERNIONS.

By *P. G. Tait, M.A.*

(*Continued from page* 91.)

XV. Before commencing the set of examples promised, it may be useful to devote a page or two to the interpretation of various quaternion expressions, especially as this preparation will be of great assistance in understanding the examples themselves.

Thus, by IX. (5), we have

$$S\alpha\beta = S\beta\alpha.$$

By XIII. we see that

$$\begin{aligned} S\alpha\beta &= S.\alpha\,(T\beta\cos\theta\,U\alpha + T\beta\sin\theta\,U\gamma\,U\alpha), \\ &= T\beta\cos\theta\,S.\alpha U\alpha\ (\text{X. (4)}), \\ &= T\beta\cos\theta\,\frac{S\alpha\alpha}{T\alpha}, \\ &= -\,T\alpha T\beta\cos\theta\ (\text{IX. (2)})\ \ldots\ldots\ldots\ldots\ldots\ldots(1). \end{aligned}$$

Hence *the scalar of the product of any two vectors is the rectangle under their lengths multiplied by the cosine of their inclination, and taken negatively*, and therefore if $S.\alpha\beta = 0$, $\alpha$ is perpendicular to $\beta$.

By a similar process,

$$V\alpha\beta = T\alpha T\beta\sin\theta.U\gamma\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

and therefore evidently,

$$U.V\alpha\beta = U\gamma,$$

$$T.V\alpha\beta = T\alpha T\beta\sin\theta.$$

Hence *the vector of the product of any two vectors is another vector perpendicular to their plane* (compare V.) *whose length is the product of their lengths multiplied by the sine of their inclination.* Also $T.V\alpha\beta$ is double the area of the triangle, two of whose sides are $\alpha$, $\beta$.

Again, $$S.\alpha\beta\gamma = S.\alpha V\beta\gamma.$$

Let $$V.\beta\gamma = T\beta T\gamma\sin\phi\,U\delta \text{ by } (2);$$

therefore $S.\alpha\beta\gamma = T\alpha T\beta T\gamma \sin\phi S.U\alpha U\delta$

$$= - T\alpha T\beta T\gamma \sin\phi\alpha \cos\hat{\delta} \ldots\ldots\ldots\ldots (3),$$

if we represent by $\widehat{\alpha\delta}$ the angle between the directions of $\alpha$ and $\delta$.

Hence $S.\alpha\beta\gamma = 0$

shows that either $\sin\phi = 0$, *i.e.*, $\beta \parallel \gamma$, *i.e.*, $V.\beta\gamma = 0$, (or, of course, $\gamma \parallel \alpha$, or $\alpha \parallel \beta$) or, finally,

$$\cos\widehat{\alpha\delta} = 0;$$

therefore $\delta$, which is perpendicular to $\beta$ and $\gamma$, is also perpendicular to $\alpha$. Hence *if* $S.\alpha\beta\gamma = 0$, $\alpha$, $\beta$, $\gamma$ *are in one plane.*

It is evident from the general expression (3) that $S.\alpha\beta\gamma$ is six times the volume of the tetrahedron whose conterminous edges are $\alpha$, $\beta$, $\gamma$. Thus for the volume of the tetrahedron one of whose vertices is the origin, and whose other vertices are at $x_1 y_1 z_1$, $x_2 y_2 z_2$, $x_3 y_3 z_3$, we have

$$\text{vol.} = \tfrac{1}{6} S (x_1 i + y_1 j + z_1 k)(x_2 i + \&c.)(x_3 i + \&c.)$$

$$= -\tfrac{1}{6} \begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix}, \text{ a well-known result.}$$

XVI. It is evident that all of these formulæ contain propositions in spherical trigonometry.

Thus if $\alpha$, $\beta$, and $\gamma$ be unit vectors to $A$, $B$, $C$ the angles of a spherical triangle; and if $A'$, $B'$, $C'$ be the poles of the sides

$$S\alpha\beta\gamma = S\beta\gamma\alpha = S\gamma\alpha\beta$$

may be interpreted thus,

$$\sin BC \cos AA' = \sin CA \cos BB' = \sin AB \cos CC'.$$

Or, to take another case, given by Sir W. R. Hamilton in his great work (p. 529),

$$S.V\alpha\beta V\beta\gamma = S.\alpha V.\beta V\beta\gamma$$
$$= S.\alpha (\beta^2\gamma - \beta S\beta\gamma)$$
$$= \beta^2 S\alpha\gamma - S\alpha\beta S\beta\gamma,$$

which, if $\alpha$, $\beta$, $\gamma$ be the unit-vectors from the centre of the sphere to the angles $A$, $B$, $C$ of a spherical triangle, is easily seen to be equivalent to

$$+\sin c \sin a \cos B = \cos b - \cos c \cos a,$$

the fundamental formula of spherical trigonometry.

An endless series of interesting results may be obtained by similar interpretations of quaternion formulæ; as instances we leave to the reader the following:

$$V.V\alpha\beta V\beta\gamma = -\beta S\alpha\beta\gamma,$$
$$V.V\alpha\beta V\gamma\delta = \alpha S\beta\gamma\delta - \beta S\gamma\delta\alpha,$$
$$S.\alpha V\alpha\beta V\beta\gamma = -S\alpha\beta S\alpha\beta\gamma,$$
$$S.\alpha V.\alpha\beta V\beta\gamma = 0.$$

XVII. ($a$) Let us now consider generally the equation to the straight line.

Let $O$ be the origin, $AB$ the line; $A$ a fixed, $P$ a variable point in it. $OA = \alpha$, $P = \rho$, $AB = \beta$. Then evidently

$$\rho - \alpha = AP = x\beta,$$

or

$$\rho = \alpha + x\beta,$$

$x$ being a number.

This may evidently be written

$$V.\beta(\rho - \alpha) = 0,$$

and this transformation is worthy of note, as shewing us *one* method of eliminating a scalar from a vector equation.

($b$) The equation to the line joining $A$ and $B$, $OA = \alpha$, $OB = \beta$, is $\rho = \alpha + x(\beta - \alpha)$, or $\rho = \beta + y(\beta - \alpha)$, which are evidently the same, as we find by putting $1 + y$ for $x$.

($c$) Any two parallel lines have equations such as

$$\left.\begin{aligned}\rho &= \alpha + x\beta,\\ \rho &= \alpha_1 + x_1\beta.\end{aligned}\right\}$$

($d$) A line through $O$ is $\rho \parallel \beta$, or $V.\rho\beta = 0$, or $\rho = x\beta$.

($e$) The line through $A$ perpendicular to $OA$ and $OB$ has evidently the equation

$$\rho = \alpha + xV\alpha\beta,$$

and so on.

($f$) The shortest line from $O$ to $AB$ is found thus:

Equation to $AB$ is $\rho = \alpha + x\beta$.

Let $T\rho$ be a minimum, then

$$\frac{d}{dx} T(\alpha + x\beta) = 0,$$

when $\alpha + x\beta$ is the shortest vector required. This leads to

$$S.\beta(\alpha + x\beta) = 0,$$

*i.e.*, the shortest line is, as we know, perpendicular to $AB$.

[To obtain the above result, or to differentiate $T\rho$ generally, we have

$$T\rho^2 = -\rho^2 = -\rho\rho;$$

therefore $$2T\rho dT\rho = -d\rho\rho - \rho d\rho \text{ (XIV.)}$$
$$= -2S\rho d\rho \text{ (IX. 5')};$$

therefore $$dT\rho = -\frac{S\rho d\rho}{T\rho}.]$$

($g$) Given two lines

$$\left.\begin{aligned} \rho &= \alpha + x\beta, \\ \rho_1 &= \alpha_1 + x_1\beta_1, \end{aligned}\right\}$$

find the shortest vector distance between them.

If this be $\delta$, we have

$$T\delta = T(\rho - \rho_1),$$

and differentiating for the minimum,

$$0 = S(\rho - \rho)(d\rho - d\rho_1).$$

As $d\rho$ and $d\rho_1$ are independent, this gives

$$S(\rho - \rho_1)\,d\rho = 0,$$
$$S(\rho - \rho_1)\,d\rho_1 = 0.$$

But $$d\rho = \beta dx, \quad d\rho_1 = \beta_1 dx_1;$$

therefore $$\left.\begin{aligned} S\beta(\rho - \rho_1) &= 0, \\ S\beta_1(\rho - \rho_1) &= 0, \end{aligned}\right\}$$

or the required line is perpendicular to both the given lines. It is therefore parallel to $V\beta\beta_1$.

Hence $\rho - \rho_1 = \alpha + x\beta - \alpha_1 - x_1\beta_1 = yV.\beta\beta_1$.........(1).

Operate by $S.\beta\beta_1$, that is, multiply by $\beta\beta_1$ and take the scalar of the product, and we get

$$S.\beta\beta_1(\alpha-\alpha_1)=-yT^2.V\beta\beta_1.$$

Hence the shortest distance is

$$yT.V\beta\beta_1=-\frac{S.\beta\beta_1(\alpha-\alpha_1)}{T.V\beta\beta_1}=-S(\alpha-\alpha_1)\,UV\beta\beta_1.$$

And, by operating on (1) by $S.\beta$ and $S.\beta_1$, we get $x$ and $x_1$, and thus the extremities of the line are found at once.

(*h*) The perpendicular from $C\,(OC=\gamma)$ upon $\rho=\alpha+x\beta$ is easily found. Let it be $\delta$, then

$$\gamma+\delta=\alpha+x\beta,\ \text{and}\ S\beta\delta=0.$$

Operate by $S.\beta$, and we get

$$x=\frac{S\beta(\gamma-\alpha)}{\beta^2}.$$

Hence
$$\begin{aligned}\delta&=\beta^{-1}S\beta(\gamma-\alpha)-(\gamma-\alpha)\\&=-\beta^{-1}V\beta(\gamma-\alpha),\end{aligned}$$

by an obvious transformation.

(*i*) The expression $\alpha^{-1}\beta\alpha$ represents a vector since it has obviously no scalar part. Let it be written $\gamma$. Then

$$S\alpha\gamma=S\alpha\beta,\quad V\alpha\gamma=-V\alpha\beta,$$

and
$$T\gamma=T\beta.$$

From these it follows that $\beta$ and $\gamma$ are equally inclined to $\alpha$, on opposite sides of it, or if $\beta$ be an incident ray of light, $\alpha$ the normal to a reflecting surface, $\gamma$ or $\alpha^{-1}\beta\alpha$ is the reflected ray.

Similarly $\alpha^{-m}\beta\alpha^m$ is a vector derived from $\beta$ by a conical rotation about $\alpha$, through an angle equal to $m\pi$.

XVIII. It is to be observed that the equation to a straight line (being a *vector* one)

$$\rho=\alpha+x\beta,$$

is equivalent to *three* scalar equations. But as these involve the indeterminate quantity $x$, they are really equivalent to *two* only as in Cartesian Geometry.

It is time now to consider the quaternion equation to a plane, and to show how two planes determine a line, &c. &c.

XIX. (*a*) $S.\alpha\rho = 0$ shows that $\rho$ is perpendicular to $\alpha$. The extremity of $\rho$ is therefore always in the plane through the origin perpendicular to $\alpha$, or $S\alpha\rho = 0$ *is the equation to a plane through the origin.*

Similarly $S.\alpha(\rho - \beta) = 0$ shows that $\rho - \beta$ is perpendicular to $\alpha$; *i.e.*, if $\rho = OP$ and $\beta = OB$, $BP$ is always perpendicular to $\alpha$. Hence the locus of $P$ (the extremity of $\rho$) is a plane passing through $B$, and perpendicular to $\alpha$.

(*b*) *To find the vector perpendicular from B upon* $S.\alpha\rho = 0$. It is evidently parallel to $\alpha$, let it be $x\alpha$, then one value of $\rho$ is $\beta + x\alpha$. Hence

$$S\alpha(\beta + x\alpha) = 0, \text{ or } x = -\frac{S\alpha\beta}{\alpha^2},$$

and the vector perpendicular is therefore $= -\alpha^{-1}S\alpha\beta$.

(*c*) *The plane which contains A, B, and C* may be thus found. If $P(OP = \rho)$ be any point in it, $\rho - \alpha$, $\alpha - \beta$, and $\beta - \gamma$ are vectors parallel to it. Hence

$$S.(\rho - \alpha)(\alpha - \beta)(\beta - \gamma) = 0,$$

that is, $$S\rho(V\alpha\beta + V\beta\gamma + V\gamma\alpha) - S\alpha\beta\gamma = 0.$$

Hence, if $$\delta = x(V\alpha\beta + V\beta\gamma + V\gamma\alpha)$$

be the perpendicular from $O$ on the plane containing $A$, $B$, and $C$,

$$\delta = (V\alpha\beta + V\beta\gamma + V\gamma\alpha)^{-1} S.\alpha\beta\gamma.$$

The geometrical interpretation of these formulæ is easy, and gives in a very simple manner some remarkable properties of a tetrahedron. We leave it to the reader.

(*d*) The plane passing through $O$ and the line of intersection of $S\alpha\rho = a$ and $S\beta\rho = b$ has the form

$$S.\rho(a\beta - b\alpha) = 0,$$

for, 1st, this is a plane; 2nd, it passes through $O$; 3rd, if $\rho$ belong to $S\alpha\rho = a$, it also belongs to $S\beta\rho = b$.

(*e*) In the simple case of two planes through the origin

$$\left.\begin{aligned}S\alpha\rho&=0,\\ S\alpha_1\rho&=0,\end{aligned}\right\}$$

we have $\rho = xV\alpha\alpha_1$ for the intersection.

We leave to the reader the investigation for *any* two planes

$$\left.\begin{aligned}S\alpha(\rho-\beta)&=0,\\ S\alpha_1(\rho-\beta_1)&=0.\end{aligned}\right\}$$

(*f*) The equation to the plane which contains the lines

$$\rho = x\alpha,$$

$$\rho = y\beta,$$

is evidently $$S.\alpha\beta\rho = 0 \text{ (XV.)}.$$

(*g*) The equation to a plane passing through

$$\rho = x\alpha,$$

and perpendicular to $$S\beta\rho = 0,$$

is evidently $$S.\alpha\beta\rho = 0.$$

(*h*) The plane through a point $A$ ($OA = \alpha$), perpendicular to the line of intersection of

$$S\beta\rho = 0, \text{ and } S\gamma\rho = 0$$

is evidently $$S.\beta\gamma(\rho - \alpha) = 0.$$

(*i*) A plane parallel to

$$\rho = \alpha + x\beta,$$

and $$\rho = \alpha_1 + x_1\beta_1,$$

is evidently $$S.\beta\beta_1(\rho - \delta) = 0,$$

where $\delta$ is the vector of any point in it.

Now $\delta = \alpha$ if the plane contain the first line, $= \alpha_1$ if it contain the second.

Hence if $yV\beta\beta_1$ be the vector shortest distance between the lines,

$$S.\beta\beta_1(\alpha - \alpha_1 - yV\beta\beta_1) = 0,$$

or $$yT.V\beta\beta_1 = -S(\alpha - \alpha_1)\,UV\beta\beta_1,$$

a result already obtained in a different way (XVII. (*g*)).

(*j*) The plane through $A$ and $B$ ($OA=\alpha$, $OB=\beta$) and perpendicular to $S.\gamma\rho=0$, is parallel to $\beta-\alpha$ and to $\gamma$, and its equation is therefore of the form

$$S.(\beta-\alpha)\gamma(\rho-\delta)=0,$$

$\delta$ being as yet undetermined. But as it belongs to *any* definite point in the plane, we may put for it $\alpha$ or $\beta$, and either substitution gives us the required result, viz.,

$$S.(\beta-\alpha)\gamma\rho=S\alpha\beta\gamma.$$

XX. It appears from these results, although we have not attempted to give them in the most concise manner, but have simply put down the investigation in what seemed its most direct form, that the quaternion calculus is capable of dealing with questions about lines and planes in the most satisfactory manner. We will now give one or two examples of a more difficult character; and then go on to the consideration of surfaces of the second order, the method of treatment for which will be first explained.

*Find the equation to that plane passing through the origin which makes equal angles with three given lines. Also find the angles in question.*

Let $\alpha$, $\beta$, $\gamma$ be unit vectors in the directions of the lines. Then, evidently, if $S\delta\rho=0$ be the equation to the plane, we have

$$S\alpha\delta=S\beta\delta=S\gamma\delta=x,$$

suppose, where $\frac{-x}{T\delta}$ is the sine of each of the required angles.

By XI. (5) we see at once that

$$\delta S\alpha\beta\gamma=x(V\alpha\beta+V\beta\gamma+V\gamma\alpha).$$

Hence $$S.\rho(V\alpha\beta+\&c.)=0$$

is the required equation, and the required sine is

$$\frac{S.\alpha\beta\gamma}{T(V\alpha\beta+\&c.)}.$$

*Find the locus of the middle points of a series of straight lines each parallel to a given plane and having its extremities in two fixed lines.*

Let $S\alpha\rho=0$ be the plane,

$$\rho=\beta+x\gamma,\quad \rho=\beta_1+y\gamma_1,$$

the fixed lines, and let $x$ and $y$ correspond to the extremities of one of the variable lines, $\varpi$ being the vector of its middle point. Then, obviously,

$$2\varpi = \beta + x\gamma + \beta_1 + y\gamma_1.$$

Also $$S.\alpha(\beta - \beta_1 + x\gamma - y\gamma_1) = 0.$$

This gives a linear relation between $x$ and $y$, hence, if we substitute for the latter in the preceding equation, we obtain a result of the form

$$\varpi = \delta + x\epsilon,$$

where $\delta$ and $\epsilon$ are known vectors. The locus is therefore a straight line.

*Given any number of points $A$, $B$, $C$, &c., whose vectors are $\alpha_1$, $\alpha_2$, $\alpha_3$, &c., find the plane through the origin for which the sum of the squares of the perpendiculars let fall from these points is a minimum.*

Let $S\varpi\rho = 0$ be the required equation, and suppose, as we evidently may, $T\varpi = 1$.

Then the perpendiculars are $\varpi^{-1}S\varpi\alpha$, &c. Hence $\Sigma S^2\varpi\alpha$ is a maximum. Hence, differentiating,

$$\left.\begin{aligned} \Sigma\, S\varpi\alpha\, S\varpi'\alpha &= 0 \\ \text{Also} \qquad T\varpi = 1 \text{ gives } S\varpi\varpi' &= 0 \end{aligned}\right\}.$$

Hence, as $\varpi'$ may have any of an infinite number of values, we find to determine $\varpi$

$$\Sigma.\alpha S\varpi\alpha = x\varpi,$$

a vector equation of the first degree. Or

$$\Sigma\, V\alpha\varpi\, S\alpha\varpi = 0.$$

We will interpret these expressions when we come to Conicoids.

*Of a system of three rectangular vectors passing through $O$, two are confined to given planes, find the locus of the third.*

Let them be $\varpi$, $\rho$, $\sigma$. Then

$$S\varpi\rho = S\rho\sigma = S\sigma\varpi = 0.$$

Also $$S\alpha\varpi = 0, \quad S\beta\rho = 0.$$

The problem is therefore reduced to the elimination of $\varpi$ and $\rho$ amongst these five equations. And although in general, on account of the three scalars on which a vector depends, seven scalar equations are necessary for the elimination of two vectors—yet in this case the five suffice, because the lengths of $\varpi$ and $\rho$ may be any whatever—and we may complete the formal number of equations by writing say $T\varpi = 1$, $T\rho = 1$. We will consider the process carefully, as it is of considerable importance.

$$S\alpha\varpi = 0, \quad S\sigma\varpi = 0,$$

hence $$\varpi = x V\alpha\sigma.$$

$$S\beta\rho = 0, \quad S\sigma\rho = 0;$$

therefore $$\rho = y V\beta\sigma.$$

Substitute these values in the remaining equation

$$S\varpi\rho = 0,$$

and we get $$S . V\alpha\sigma V\beta\sigma = 0,$$

or $$S\alpha\sigma S\beta\sigma - \sigma^2 S\alpha\beta = 0,$$

an equation of the second degree.

XXI. This leads us naturally to the consideration of equations of a degree higher than the first. As that of the sphere is far simpler than the others, we will consider it first.

$T\rho = T\alpha$ expresses that the *length* of $OP$ is constant and equal to that of $OA$. Hence $P$ lies anywhere on the sphere whose centre is origin and radius $OA$. This may be written in the form

$$\rho^2 = \alpha^2.$$

If the centre be at $A$ and the radius be equal to the length of $OB$, we have, for the equation,

$$T(\rho - \alpha) = T\beta,$$

or $$(\rho - \alpha)^2 = \beta^2.$$

If the origin be on the surface $T\alpha = T\beta$, and the equation is

$$(\rho - \alpha)^2 = \alpha^2,$$

or $$\rho^2 - 2S\alpha\rho = 0.$$

The latter equation may be written

$$S.\rho(\rho - 2\alpha) = 0.$$

That is, if $O'$ be the other extremity of the diameter passing through $O$, and $OP=\rho$, we have $OP$ perpendicular to $O'P$, a well-known property.

For the intersection of two spheres

$$\rho^2-2S\alpha\rho+\alpha^2=\beta^2,$$

and

$$\rho^2-2S\alpha_1\rho+\alpha_1^2=\beta_1^2,$$

subtracting the equations, we get

$$2S(\alpha-\alpha_1)\rho=(\alpha^2-\alpha_1^2)-(\beta^2-\beta_1^2),$$

the equation to a plane perpendicular to $\alpha-\alpha_1$ the vector joining the centres.

*Find the locus of the feet of perpendiculars from the origin on planes passing through a given point.*

If $S\varpi(\rho-\alpha)=0$ be one of the planes, the vector perpendicular is

$$\rho=\varpi^{-1}S\alpha\varpi.$$

Now this vector equation is equivalent to three scalar equations, and therefore $\varpi$ can be eliminated from it, since its tensor is not involved and it is thus only equivalent to two scalar unknowns.

It gives

$$\rho^2=\varpi^{-2}S^2.\alpha\varpi,$$

also

$$S\alpha\rho=\frac{S^2\alpha\varpi}{\varpi^2}.$$

Hence

$$\rho^2=S\alpha\rho,$$

or

$$T\left(\rho-\frac{\alpha}{2}\right)=T\frac{\alpha}{2},$$

the sphere as it ought to be.

*Find the locus of a point the ratio of whose distances from two given points is constant.*

Let them be $O$ and $A$, $OA=\alpha$. At once

$$T(\rho-\alpha)=nT\rho,$$

$$\rho^2-2S\alpha\rho+\alpha^2=n^2\rho^2;$$

therefore

$$T\left(\rho-\frac{\alpha}{1-n^2}\right)=T\left(\frac{n\alpha}{1-n^2}\right).$$

A sphere of which the radius is $T\left(\frac{n\alpha}{1-n^2}\right)$ and the centre is at $B$, where $OB=\frac{\alpha}{1-n^2}$, a definite point in the line $OA$.

XXII. Generally, if $F(\rho)=0$, a scalar equation, be the equation to a surface, and if its differential be

$$S.\nu d\rho=0,$$

(where $\nu$ is some vector depending on $\rho$ and the form of $F$), $\nu$ is the *normal* vector. For it is perpendicular to $d\rho$ which is in the direction of any indefinitely short vector-chord joining the extremity of $\rho$ with a contiguous point on the surface.

The equation to the tangent-plane to the surface at the extremity of $\rho$ is, therefore,

$$S.\nu(\varpi-\rho)=0,$$

where $\varpi$ is the vector of any point in the plane.

Applying this to the sphere

$$T(\rho-\alpha)=T\beta,$$

we get
$$S.(\rho-\alpha)\,d\rho=0.$$

Hence the normal is $(\rho-\alpha)$, the vector joining any point with the centre. And the tangent plane is

$$S(\rho-\alpha)(\varpi-\rho)=0,$$

or
$$S.\varpi(\rho-\alpha)=\rho^2-S\alpha\rho=S\alpha\rho-\alpha^2+\beta^2.$$

This may be written

$$S.(\varpi-\alpha)(\rho-\alpha)=\beta^2.$$

To simplify matters, but without losing any useful generality, let us consider a sphere with $O$ as centre. Then

$$\rho^2=\beta^2,$$

and the tangent plane is

$$S\varpi\rho=\beta^2.$$

If this pass through a point $C$ $(OC=\gamma)$ we have

$$S\gamma\rho=\beta^2,$$

a plane which intersects the sphere in all points where planes through $C$ can touch the sphere, in other words the polar plane of $C$.

The perpendicular from the centre on this plane is evidently $\frac{T\beta^2}{T\gamma}.U\gamma$, which agrees with long known results.

Let a line be drawn parallel to $\alpha$ from $C$, cutting the sphere, its equation is

$$\rho = \gamma + x\alpha,$$

and the sphere is $$\rho^2 = \beta^2.$$

Hence for the intersections

$$\gamma^2 + 2xS\alpha\gamma + x^2\alpha^2 = \beta^2;$$

therefore $$x^2 + 2x\frac{S\alpha\gamma}{\alpha^2} + \left(\frac{S\alpha\gamma}{\alpha^2}\right)^2 = \frac{\beta^2 - \gamma^2}{\alpha^2} + \frac{(S\alpha\gamma)^2}{\alpha^4},$$

$$x = -\frac{S\alpha\gamma}{\alpha^2} \pm \frac{1}{\alpha^2}\sqrt{(\beta^2\alpha^2 + V^2.\alpha\gamma)},$$

which gives the two values of $x$.

When these are equal the line is a tangent. Hence

$$\beta^2\alpha^2 = T^2 V\alpha\gamma,$$

which is therefore the equation to a cone similar and similarly situated to the tangent cone from $C$, but having its vertex at $O$. That it does represent a cone is evident from the fact that the above equation is satisfied whatever be the tensor of $\alpha$, provided its direction be suitable.

In XX. we arrived at the equation ($\sigma$ being the vector of any point in the surface)

$$S\alpha\sigma\, S\beta\sigma - \sigma^2 S\alpha\beta = 0,$$

which similar reasoning shows to represent a cone. To find what cone, suppose it cut by a plane perpendicular to $\alpha$, *i.e.*, $S\alpha\sigma = a$, and we have as the equation to another surface passing through the line of intersection

$$\sigma^2 - \frac{a}{S\alpha\beta} S\beta\sigma = 0,$$

a sphere. Hence any section of the surface in question made perpendicular to $\alpha$, and the same is true of $\beta$, is a circle. The equation, therefore, represents an oblique cone of the second order whose circular sections are perpendicular to $\alpha$ and $\beta$.

To vary our work a little, let us seek *the envelop of a plane whose distance from the origin is constant.* Although an excessively simple case, it may be useful in giving an idea of the method of treating the question of envelops by quaternions.

Let $S.\alpha\rho = -1$ be the plane, then $\frac{1}{T\alpha}$ is the length of the perpendicular. Hence

$$T\alpha = C = \text{constant}$$

is the condition. From these equations we have, by differentiation,

$$\left.\begin{matrix} S\rho\alpha' = 0 \\ S\alpha\alpha' = 0 \end{matrix}\right\},$$

which can only be simultaneously true if

$$\alpha \parallel \rho = x\rho,$$

since $\alpha'$ may have any of an infinite set of values, therefore

$$T\alpha = xT\rho = C;$$

therefore $$x = \frac{C}{T\rho},$$

therefore $$\alpha = x\rho = CU\rho;$$

and, substituting in the equation to the plane,

$$T\rho = \frac{1}{C},$$

the equation to the sphere as it ought to be. We have been particularly explicit in the work of this very simple example, as it explains the method to be pursued in such cases quite as well as a more complex one.

*Perpendiculars are drawn from a point of a sphere to all tangent planes; find the locus of their extremities.*

Let $\rho^2 = \alpha^2$ be the sphere, and $A$ $(OA = \alpha)$ the point. Any tangent plane is

$$S\varpi\rho = \alpha^2,$$

and the perpendicular is parallel to $\rho$. Hence for the foot of the perpendicular

$$\varpi = \alpha + x\rho.$$

From these equations we must eliminate $\rho$ and $x$. Operate by $S.\varpi$,

$$\varpi^2 = S\alpha\varpi + x\alpha^2;$$

therefore
$$\rho = \frac{\varpi - \alpha}{x} = \frac{\alpha^2(\varpi - \alpha)}{\varpi^2 - S\alpha\varpi};$$

whence, taking the tensors, we have
$$(\varpi^2 - S\alpha\varpi)^2 = \alpha^2(\varpi - \alpha)^2,$$

the required equation; which may be written thus
$$S^2.\varpi U(\varpi - \alpha) = -\alpha^2,$$

and the interpretation of this equation gives at once a property of the solid formed by the rotation of the cardioid about its axis of symmetry.

*$A$, $B$, and $C$ are fixed points, and $AP^2 + BP^2 = CP^2$; find the locus of $P$, and explain the result when $ACB$ is a right or obtuse angle.*

Take $C$ as origin, $CA = \alpha$, $CB = \beta$, $CP = \rho$; then
$$(\rho - \alpha)^2 + (\rho - \beta)^2 = \rho^2,$$
$$\rho^2 - 2S(\alpha + \beta)\rho = -(\alpha^2 + \beta^2),$$
or
$$T\{\rho - (\alpha + \beta)\} = \sqrt{(-2S\alpha\beta)}.$$

As long as $\sqrt{(-2S\alpha\beta)}$ is possible, the locus is therefore a sphere, whose centre is at the extremity of the vector $\alpha + \beta$, found by completing the parallelogram two of whose sides are $CA$ and $CB$; when $S\alpha\beta = 0$, *i.e.*, when $ACB$ is a right angle, the locus is the point $\rho = \alpha + \beta$; and when $S\alpha\beta$ is positive, or $ACB$ obtuse, there is no point which satisfies the condition.

*A sphere touches each of two straight lines which do not meet; find the locus of its centre.*

Let the lines be
$$\left.\begin{array}{l} \rho = \alpha + x\beta \\ \rho = -\alpha + x_1\beta_1 \end{array}\right\},$$

where we may evidently suppose $\alpha$ perpendicular to $\beta$ and $\beta_1$.

Let $\sigma$ be the vector of the centre, and $\rho$ the vector of any point, of one of the spheres, $r$ being its radius; the equation to the sphere is
$$T(\rho - \sigma) = r.$$

The lines must each intersect this in two coincident points;

hence, substituting, the following equations should give pairs of equal values for $x$ and $x_1$ respectively,

$$T(\alpha + x\beta - \sigma) = r,$$
$$T(-\alpha + x_1\beta_1 - \sigma) = r.$$

From the first, since $S\alpha\beta = 0$,

$$x^2\beta^2 - 2xS\beta\sigma + (\alpha - \sigma)^2 = -r^2,$$

and the condition of equal roots is

$$S^2\beta\sigma = \beta^2\{(\alpha - \sigma)^2 + r^2\}.$$

A similar condition is derived from the second, and by the elimination of $r$ between them we get the required equation

$$\beta^{-2}S^2\beta\sigma - \beta_1^{-2}S^2\beta_1\sigma = (\alpha - \sigma)^2 - (\alpha + \sigma)^2$$
$$= -4S\alpha\sigma.$$

To show the ease of passing to coordinate geometry when the quaternion form of an equation may not from our want of familiarity with it at once suggest the interpretation; let us suppose the axes (not rectangular) to be parallel to $\beta$, $\beta_1$, and $\alpha$. The above takes at once the form

$$x^2 - y^2 = pz,$$

and we see that the required locus is the hyperbolic paraboloid.

As an exercise, we may notice that the above quaternion equation may easily be seen to be equivalent to

$$TV.(\alpha - \sigma)\,U\beta = TV.(\alpha + \sigma)\,U\beta_1.$$

Simple as the above mode of solution may appear, it will be evident, on looking at this last form of the required equation, that it might have been written down at once from a consideration of the problem, the two members merely expressing the lengths of the perpendiculars from the extremity of $\sigma$ upon the two given lines.

*Two spheres being given, show that spheres which cut these at given angles cut at right angles another fixed sphere.*

If $\varpi$ be the vector joining the centres of two spheres whose radii are $a$ and $b$, it is evident that the cosine of the angle of intersection is

$$\frac{\varpi^2 + a^2 + b^2}{2ab}.$$

Hence $\alpha$, $\alpha_1$, and $\rho$ being the vectors of the centres, and $a$, $a_1$, and $r$ being the radii, of the two fixed and one of the variable spheres, $A$ and $A_1$ the angles of intersection,

$$(\rho-\alpha)^2+a^2+r^2=2ar\cos A,$$

$$(\rho-\alpha_1)^2+a_1^2+r^2=2a_1r\cos A_1.$$

Eliminating the first power of $r$ we evidently must obtain a result such as

$$(\rho-\beta)^2+b^2+r^2=0,$$

where $\beta$ is the vector of the centre, and $b$ the radius, of a fixed sphere which is cut at right angles by all the variable ones. By effecting the elimination it is easy to find $b$ and $\beta$ in terms of given quantities.

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# ON TETRAHEDRONS.

By *A. F. Torry, B.A.*, Scholar of St. John's College, Cambridge.

THE following construction is intended to exhibit geometrically some of the well known theorems concerning tetrahedrons.

Let $ABCD$ (fig. 28) be any tetrahedron: through each pair of opposite edges draw two parallel planes: the tetrahedron is thus enclosed in a parallelopiped the diagonals of whose opposite faces are opposite edges of the tetrahedron. Let the angular points of this solid opposite to $A$, $B$, $C$, $D$ be respectively $\alpha$, $\beta$, $\gamma$, $\delta$.

The following three propositions are self-evident.

(I.) The straight lines joining the middle points of opposite edges of the tetrahedron pass through the centre of the parallelopiped and are parallel and equal to its edges.

(II.) If these lines are at right angles to one another the parallelopiped is rectangular.

(III.) If opposite edges of the tetrahedron are at right angles to one another the faces of the parallelopiped are rhombic.

PROP. *The volumes of the corners of the parallelopiped cut off by the faces of the tetrahedron are equal.*

For the figure $DAB\gamma = \frac{1}{2}DA\delta B\gamma =$ one-sixth of the parallelopiped, and similarly for the others.

COR. *The volume of the tetrahedron is one-third of that of the parallelopiped.*

If $p$ be the shortest distance between $AB$, $CD$, $\theta$ the angle between these lines; the area of the face

$$\gamma\delta = \tfrac{1}{2}AB\,.\,CD \sin\theta,$$

and the volume of the parallelopiped

$$= \tfrac{1}{2}AB\,.\,CDp \sin\theta,$$

hence the volume of the tetrahedron

$$= \tfrac{1}{6}AB\,.\,CDp \sin\theta.$$

(Hymers' *Geometry of Three Dimensions*, p. 213.)

PROP. *The centre of gravity of the tetrahedron is the same as that of the parallelopiped.*

Consider the coordinates of the centre of gravity of the four corners referred to $A\beta$, $A\gamma$, $A\delta$, as axes: the abscissa measured along $A\beta$ is one-fourth of that edge for the pyramids $DA\gamma B$, $CA\delta B$ and three-fourths of the edge for the other two corners, the volumes also of the corners are equal, therefore the abscissa of the centre of gravity of the four is $\frac{1}{2}A\beta$,—the same as for the parallelopiped.

Thus the centre of gravity of the corners, and therefore that of the tetrahedron, coincides with that of the parallelopiped. (See also Ex. 5, p. 74 in Frost and Wolstenholme's *Solid Geometry*.)

PROP. *If the lines joining the middle points of opposite edges of the tetrahedron are at right angles, the opposite edges are equal.*

This follows immediately from (II.), since the diagonals of a rectangle are equal.

The faces of the tetrahedron in this case are equal in every respect.

**Prop.** *The sum of the squares of the edges of the tetrahedron is equal to four times that of the lines joining the middle points of the opposite edges.*

The squares of the diagonals of a parallelogram being together equal to those of its sides, the sum of the squares of the edges of the tetrahedron is equal to four times that on three adjacent edges of the parallelopiped, from which, by (I.), the proposition follows.

**Prop.** *If the edges of the tetrahedron which terminate in* $D$ *be* $a$, $b$, $c$; *and the edges respectively opposite* $a'$, $b'$, $c'$; *and if* $\theta$ *be the angle between* $DA$, $BC$, *then*

$$\cos\theta = \frac{b^2 + b'^2 \sim (c^2 + c'^2)}{2aa'}.$$

For $b^2 + b'^2 - (c^2 + c'^2) = 2\,(A\delta^2 + A\gamma^2) - 2\,(A\beta^2 + A\delta^2)$,

$$= 2\,(A\gamma^2 - A\beta^2),$$

and $$\cos\theta = \frac{A\gamma^2 \sim A\beta^2}{\beta\gamma . AD},$$

therefore, &c. Q.E.D.

**Prop.** *If two pairs of opposite edges be respectively at right angles, the third pair will also be at right angles to each other.*

By (III.), this is the same as *if two pairs of opposite faces of the parallelopiped be rhombic the third is so also.*

But if $A\delta = B\delta$ and $B\delta = C\delta$, then $A\delta = C\delta$. Q.E.D. (*Quarterly Journal*, Vol. III., pp. 89 and 145.)

Other examples might be given, the proofs of which are simplified by this method: the above will sufficiently illustrate the method to be adopted in any case that may occur.

# PROPERTY OF CONIC SECTIONS.

By the Rev. *N. M. Ferrers, M.A.*, Fellow and Mathematical Lecturer of Gonville and Caius College, Cambridge.

THE property of the rectangular hyperbola mentioned by Mr. Cayley, on p. 77, leads, when reciprocated, to the following result: If a system of conics touch four given straight lines, their director circles (*i.e.*, the circles which are the loci of the intersection of two tangents at right angles to one another) have a common radical axis. For considering any two rectangular hyperbolas, and reciprocating them with respect to any point $O$, we obtain two conics, the pair of tangents drawn to each of which from $O$ are at right angles to each other. Hence, from the property of the rectangular hyperbola above referred to, it follows that every conic touching the four common tangents to the two given ones will have the tangents drawn to it from $O$, at right angles to each other. From this may be easily deduced the theorem.

A direct investigation of this property will, perhaps, be not without interest. The following demonstration shews that the common radical axis is the directrix of the parabola which touches the four given straight lines. The coordinates employed are *triangular*.

Let the diagonals of the quadrilateral, formed by the four straight lines, be taken as lines of reference. Then, if the equation of one of the straight lines be

$$l\alpha + m\beta + n\gamma = 0,$$

those of the others will be

$$\begin{aligned} -l\alpha + m\beta + n\gamma &= 0, \\ l\alpha - m\beta + n\gamma &= 0, \\ l\alpha + m\beta - n\gamma &= 0. \end{aligned}$$

And the general equation of a conic touching these four straight lines will be

$$u\alpha^2 + v\beta^2 + w\gamma^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$u$, $v$, $w$ being connected by the relation

$$l^2vw + m^2wu + n^2uv = 0 \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

If the conic be a parabola, we have also

$$vw + wu + uv = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

And the focus of the parabola touching (1) will be found (by adapting to triangular coordinates Mr. Whitworth's equations on p. 98) to be given by the equations,

$$\frac{vw}{a^2}(2\alpha - 1) = \frac{wu}{b^2}(2\beta - 1) = \frac{uv}{c^2}(2\gamma - 1),$$

whence

$$\alpha : \beta : \gamma : 1 :: b^2v + c^2w : c^2w + a^2u : a^2u + b^2v : 2(a^2u + b^2v + c^2w).$$

The equation of the directrix of this parabola will, therefore, be

$$(b^2v + c^2w)\, u\alpha + (c^2w + a^2u)\, v\beta + (a^2u + b^2v)\, w\gamma = 0,$$

which, substituting for $vw$, $wu$, $uv$ their values given by (2) and (3), becomes

$$\{b^2(l^2 - m^2) + c^2(n^2 - l^2)\}\,\alpha + \{c^2(m^2 - n^2) + a^2(l^2 - m^2)\}\,\beta$$
$$+ \{a^2(n^2 - l^2) + b^2(m^2 - n^2)\}\,\gamma = 0 \ldots\ldots\ldots\ldots (4).$$

Again, the two tangents drawn to (1) from any point $(f, g, h)$ are given by the equation

$$(uf^2 + vg^2 + wh^2)(u\alpha^2 + v\beta^2 + w\gamma^2) - (uf\alpha + vg\beta + wh\gamma)^2 = 0,$$

or $(uvg^2 + wuh^2)\,\alpha^2 + (vwh^2 + uvf^2)\,\beta^2 + (wuf^2 + vwg^2)\,\gamma^2$

$$- 2vwgh\beta\gamma - 2wuhf\gamma\alpha - 2uvfg\alpha\beta = 0.$$

If these be at right angles to one another, we must have,

$$(uvg^2 + wuh^2)\,a^2 + (vwh^2 + uvf^2)\,b^2 + (wuf^2 + vwg^2)\,c^2$$
$$+ vwgh(b^2 + c^2 - a^2) + wuhf(c^2 + a^2 - b^2) + uvfg(a^2 + b^2 - c^2) = 0,$$

or $(b^2v + c^2w)\,uf^2 + (c^2w + a^2u)\,vg^2 + (a^2u + b^2v)\,wh^2$

$$+ (b^2 + c^2 - a^2)\,vwgh + (c^2 + a^2 - b^2)\,wuhf + (a^2 + b^2 - c^2)\,uvfg = 0,$$

which, considering $f$, $g$, $h$ as current coordinates, represents the locus of the point, the two tangents from which to (1) are at right angles to one another, that is, its director circle.

It may also be written in the form

$$vw\,\{c^2g^2+b^2h^2+(b^2+c^2-a^2)\,gh\}+wu\,\{a^2h^2+c^2f^2+(a^2+c^2-b^2)\,hf\}$$
$$+\,uv\,\{b^2f^2+a^2g^2+(b^2+a^2-c^2)fg\}=0,$$

whence, by comparison with (2), we see that the circle always passes through the points given by the equations

$$\frac{c^2g^2+b^2h^2+(b^2+c^2-a^2)\,gh}{l^2}=\frac{a^2h^2+c^2f^2+(c^2+a^2-b^2)\,hf}{m^2}$$
$$=\frac{b^2f^2+a^2g^2+(a^2+b^2-c^2)fg}{n^2}.$$

Multiplying the numerators and denominators of these fractions by $m^2-n^2$, $n^2-l^2$, $l^2-m^2$, and adding, we get

$$\{c^2\,(n^2-l^2)+b^2\,(l^2-m^2)\}f^2+\{a^2\,(l^2-m^2)+c^2\,(m^2-n^2)\}g^2$$
$$+\,\{b^2\,(m^2-n^2)+a^2\,(n^2-l^2\}\,h^2+(m^2-n^2)\,(b^2+c^2-a^2)\,gh$$
$$+\,(n^2-l^2)\,(c^2+a^2-b^2)\,hf+\,(l^2-m^2)\,(a^2+b^2-c^2)fg=0,$$

or $(f+g+h)\,[\{b^2(l^2-m^2)+c^2(n^2-l^2)\}f+\{c^2(m^2-n^2)+a^2(l^2-m^2)\}g$

$$+\,\{a^2\,(n^2-l^2)+b^2\,(m^2-n^2)\}\,h]=0,$$

shewing that the two points, through which the director circle always passes, lie on the directrix of the parabola, as might have been foreseen from geometrical considerations.

The locus of the centre of the conic which can be described touching the four given straight lines, is the straight line

$$l^2\alpha+m^2\beta+n^2\gamma=0.$$

This is necessarily perpendicular to (1). Now two rectangular hyperbolas (real or imaginary) can be described touching four given straight lines: and since, for a rectangular hyperbola, the director circle degenerates into its centre, it appears that the two points through which all the director circles pass will be real or imaginary (*i.e.*, the common radical axis will, or will not, be a chord of intersection in the strict sense of the words) according as these rectangular hyperbolas are imaginary or real. The condition that (1) should be a rectangular hyperbola is

$$a^2u+b^2v+c^2w=0,$$

and the values of $u : v : w$ obtained by combining this with (2) are imaginary or real according as

$$(la+mb+nc)\,(mb+nc-la)\,(nc+la-mb)\,(la+mb-nc)$$

is positive or negative, *i.e.*, according as each of the quantities $la$, $mb$, $nc$ (all supposed positive) is, or is not, less than the sum of the other two.

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# THE SPHERE, REFERRED TO QUADRIPLANAR COORDINATES.

By *William Allen Whitworth, B.A.*

An attempt has already been made in this Journal to display the utility of employing in certain cases equations of the form

$$\frac{\alpha-\alpha'}{\lambda}=\frac{\beta-\beta'}{\mu}=\frac{\gamma-\gamma'}{\nu}=\rho$$

to represent a straight line in trilinear coordinates. There is no difficulty in extending the investigations to three dimensions using equations of the form

$$\frac{\alpha-\alpha'}{\lambda}=\frac{\beta-\beta'}{\mu}=\frac{\gamma-\gamma'}{\nu}=\frac{\delta-\delta'}{\pi}=\rho$$

to represent a straight line in quadriplanar coordinates.

We shall simply illustrate this method by its application to the following general problems proposed in the Cambridge Mathematical Tripos Examination, Jan. 17, 1862.

1. *To find the conditions that the general equation of the second degree in quadriplanar coordinates should represent a sphere.*

Let $a$, $b$, $c$, $d$ be the angular points of the tetrahedron of reference; $p_1$, $p_2$, $p_3$, $p_4$ the perpendiculars from the angular points on the opposite faces; $A$, $B$, $C$, $D$ the areas of the faces; and let

$$f(\alpha, \beta, \gamma, \delta)=0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (1)$$

be any homogeneous equation of the second order in the quadriplanar coordinates $\alpha$, $\beta$, $\gamma$, $\delta$. Assume $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$, $\bar{\delta}$ as the coordinates of the centre of the surface represented by (1): then the equations to any diameter are

$$\frac{\alpha-\bar{\alpha}}{\lambda}=\frac{\beta-\bar{\beta}}{\mu}=\frac{\gamma-\bar{\gamma}}{\nu}=\frac{\delta-\bar{\delta}}{\pi}=\rho,$$

and the semi-diameter ($\rho$) in this direction will be given by

$$\rho^2=-\frac{f(\bar{\alpha},\ \bar{\beta},\ \bar{\gamma},\ \bar{\delta})}{f(\lambda,\ \mu,\ \nu,\ \pi)}.$$

Now if the surface be a sphere all the diameters are equal: hence $f(\lambda, \mu, \nu, \pi)$ is constant. But it will be sufficient to express the conditions that six diameters in different directions shall be equal, and it will be convenient to choose these directions parallel to the six edges of the tetrahedron. Thus, remembering that $\frac{\pm p_1}{ab}$, $\frac{\pm p_1}{ac}$ ... are the sines of the angles which the edges $ab$, $ac$ ... make with the face $A$, and attending to the changes of sign, we obtain

$$f\left(\frac{\pm p_1}{ab}, \frac{\mp p_2}{ab}, 0, 0\right)=f\left(\frac{\pm p_1}{ac}, 0, \frac{\mp p_3}{ac}, 0\right)=f\left(\frac{\pm p_1}{ad}, 0, 0, \frac{\mp p_4}{ad}\right)$$

$$=f\left(0, \frac{\pm p_2}{bc}, \frac{\mp p_3}{bc}, 0\right)=f\left(0, \frac{\pm p_2}{bd}, 0, \frac{\mp p_4}{bd}\right)=f\left(0, 0, \frac{\pm p_3}{cd}, \frac{\mp p_4}{cd}\right),$$

which five equations will be the five conditions necessary and sufficient in order that the general equation of the second degree $f(\alpha, \beta, \gamma, \delta)=0$ may represent a sphere.

II. *To find the equation to the sphere circumscribing the tetrahedron of reference.*

The general equation of a surface of the second order circumscribing the tetrahedron is

$$k\alpha\beta+l\beta\gamma+m\gamma\delta+n\delta\alpha+q\alpha\gamma+r\beta\delta=0\ldots\ldots(2).$$

The conditions that this should be a sphere are, by I.,

$$\frac{kp_1p_2}{\overline{ab}^2}=\frac{lp_2p_3}{\overline{bc}^2}=\frac{mp_3p_4}{\overline{cd}^2}=\frac{np_4p_1}{\overline{da}^2}=\frac{qp_1p_3}{\overline{ac}^2}=\frac{rp_2p_4}{\overline{ba}^2},$$

or, since $p_1, p_2, p_3, p_4$ are inversely proportional to $A, B, C, D$,

$$\frac{k}{AB\overline{ab}^2}=\frac{l}{BC\overline{bc}^2}=\frac{m}{CD\overline{cd}^2}=\frac{n}{DA\overline{da}^2}=\frac{q}{AC\overline{ac}^2}=\frac{r}{BD\overline{bd}^2}.$$

Hence, eliminating the constants in (2), the equation to the sphere circumscribing the tetrahedron of reference is

$$AB\overline{ab}^2\alpha\beta+BC\overline{bc}^2\beta\gamma+\&c.=0,$$

or, as it may be conveniently written,

$$\Sigma\left\{AB\overline{ab}^2\alpha\beta\right\}=0.$$

St. John's College, Cambridge,
*January* 17, 1862.

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# ON LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER.

By *James Cockle, M.A.*, of Trinity College, Cambridge.

(*Continued from page* 124.)

21. IF the operation $f$, performed upon $y$, be supposed to give the result $f(y)$, the analogies of algebraic notation require that we should denote by

$$\frac{1}{f},\ \text{or}\ f^{-1},$$

the operation which, performed upon $f(y)$, gives the result $y$.

22. Let $f(y)$ be given in the known or explicit form $X$, so that we have

$$f(y)=X \quad\text{........................}(31);$$

then, operating with $f^{-1}$ on each side of (31), we have

$$f^{-1}f(y)=f^0(y)=y=f^{-1}X \quad\text{..............}(32).$$

Hence if (31), treated as an equation in $y$, be soluble the form of $f^{-1}X$ is assignable. By solving (31) we express $y$ as a function of $X$, and this function is $f^{-1}X$.

23. It follows that the expression

$$\left(T\frac{d}{dx}+U\right)^{-1}X,$$

in which $T$, $U$, and $X$ are functions of $x$, is always assignable in value. For, assuming that value to be $W$, we have

$$W=\left(T\frac{d}{dx}+U\right)^{-1}X\ldots\ldots\ldots\ldots\ldots(33),$$

and, applying the operation

$$f, \text{ or } T\frac{d}{dx}+U,$$

to each side of (33), there results

$$\left(T\frac{d}{dx}+U\right)W=T\frac{dW}{dx}+UW=X,$$

a linear differential equation in $W$, whence $W$ may be determined.

24. It follows moreover that if a linear differential equation of the second order can be put under the symbolical form

$$\left(T\frac{d}{dx}+U\right)\left(t\frac{d}{dx}+u\right)y=X\ldots\ldots\ldots\ldots(34),$$

(in which $t$ and $u$, as well as $T$, $U$, and $X$, are functions of $x$) the equation is soluble. For, proceeding as above, we find

$$\left(t\frac{d}{dx}+u\right)y=\left(T\frac{d}{dx}+U\right)^{-1}X=W,$$

and

$$y=\left(t\frac{d}{dx}+u\right)^{-1}W,$$

whence $y$ is known, being in fact obtained from the linear equation

$$t\frac{dy}{dx}+uy=W.$$

25. Let the expression

$$\left(T\frac{d}{dx}+U\right]\left[t\frac{d}{dx}+u\right)$$

indicate that the bracketted terms are to be multiplied together as if $\frac{d}{dx}$ were an algebraical symbol. The product of such a multiplication is

$$Tt\left(\frac{d}{dx}\right)^2 + (Tu + Ut)\frac{d}{dx} + Uu.$$

But this is not the symbolical product which enters into the sinister of (34), and in which the order of the factors is, in general, not indifferent. For, the $\frac{d}{dx}$ in the first factor, or multiplier, being supposed to operate upon every portion of the last factor, or multiplicand, we have

$$\left(T\frac{d}{dx} + U\right)\left(t\frac{d}{dx} + u\right)$$

$$= \left(T\frac{d}{dx} + U\right]\left[t\frac{d}{dx} + u\right) + T\frac{dt}{dx}\frac{d}{dx} + T\frac{du}{dx}.$$

26. Moreover, in applying the symbolical product to $y$, we must give $\left(\frac{d}{dx}\right)^2$ the form $\frac{d^2}{dx^2}$, not $\frac{d^2}{d^2x^2}$, or $\frac{d^2}{(dx)^2}$; for

$$\left(\frac{d}{dx}\right)^2 y = \frac{d}{dx}\frac{d}{dx}y = \frac{d}{dx}\frac{dy}{dx} = \frac{d^2y}{dx^2}.$$

27. By way of example take

$$\frac{1}{x^2}\frac{d^2y}{dx^2} - \frac{1}{x^3}\frac{dy}{dx} - y = 0 \;\ldots\ldots\ldots\ldots\ldots (35),$$

which may be decomposed into

$$\left(\frac{1}{x}\frac{d}{dx} - 1\right)\left(\frac{1}{x}\frac{d}{dx} + 1\right)y = 0 \;\ldots\ldots\ldots\ldots (36).$$

Here

$$\left(\frac{1}{x}\frac{d}{dx} + 1\right)y = \left(\frac{1}{x}\frac{d}{dx} - 1\right)^{-1} 0 = e^{\frac{x^2}{2}},$$

$$y = \left(\frac{1}{x}\frac{d}{dx} + 1\right)^{-1} e^{\frac{x^2}{2}} = \frac{1}{2}e^{\frac{x^2}{2}},$$

a particular integral.

**28.** Take, again,

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-a^2y=0 \ldots\ldots\ldots\ldots (15a),$$

which may be decomposed into

$$\left\{\sqrt{(1-x^2)}\frac{d}{dx}-a\right\}\left\{\sqrt{(1-x^2)}\frac{d}{dx}+a\right\}y=0 \ldots (37),$$

and solved in a similar manner.*

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* By way of exemplifying the notation of Art. 25, write (38) in the form $Yy = 0$, $Y$ being symbolical, and suppose that

$$Y=\left(\frac{d}{dx}+U\right)\left(\frac{d}{dx}+u\right)=U_1U_2.$$

Then we pass to the symbolical decomposition of $\lambda^2 Yy = 0$ as follows:

$$\lambda^2 Y = (\lambda U_1][\lambda U_2) + \lambda^2\frac{du}{dx}$$

$$= (\lambda U_1 + \lambda v][\lambda U_2) + \lambda^2\frac{du}{dx} - \lambda v . \lambda U_2$$

$$= (\lambda U_1 + \lambda v)(\lambda U_2) - \left(\lambda v + \frac{d\lambda}{dx}\right)\lambda U_2.$$

Hence if $v$ be determined from

$$\lambda v + \frac{d\lambda}{dx} = 0, \text{ i.e. if } v = -\frac{1}{\lambda}.\frac{d\lambda}{dx},$$

we have the new decomposition

$$\lambda^2 Yy = \left(\lambda\frac{d}{dx} + \lambda U - \frac{d\lambda}{dx}\right)\left(\lambda\frac{d}{dx} + \lambda u\right)y = 0.$$

Thus, from

$$\left(\frac{d}{dx} - \frac{x}{1-x^2} - \frac{a}{\sqrt{(1-x^2)}}\right)\left(\frac{d}{dx} + \frac{a}{\sqrt{(1-x^2)}}\right)y = 0,$$

a symbolical decomposition of

$$\frac{d^2y}{dx^2} - \frac{x}{(1-x^2)}\,\frac{dy}{dx} - \frac{a^2y}{1-x^2} = 0,$$

we may deduce (37), the symbolical decomposition of (15*a*) given in Art. 28. Again,

$$\lambda Y = (U_1][\lambda U_2) + \lambda\frac{du}{dx}$$

$$= (U_1)(\lambda U_2) - \frac{d\lambda}{dx}\,\frac{d}{dx} - u\frac{d\lambda}{dx}.$$

Assume

$$(U_1)\,\mu = \frac{d\lambda}{dx}\,\frac{d}{dx} - u\frac{d\lambda}{dx},$$

29. Of these particular examples it is to be observed, 1°. that the coefficients of their last terms are constant: 2°. that it is indifferent in which order the symbolical factors are taken: 3°. that they are reducible to equations with constant coefficients by changing the independent variable from $x$ to $\int \frac{dx}{\sqrt{(P)}}$: 4°. that the reduced equations are binomial: 5°. that their two particular integrals are obtainable by reversing the order of the symbolical factors.

30. This being premised, multiply (1) into an arbitrary factor $\rho$, and putting the product under the form

$$\left(P\rho\,\frac{d^2}{dx^2} + Q\rho\,\frac{d}{dx} + R\rho\right) y = 0 \;\ldots\ldots\ldots\ldots(38),$$

assume it to be equivalent to

$$\left(T\,\frac{d}{dx} + U\right)\left(t\,\frac{d}{dx} + u\right) y = 0 \ldots\ldots\ldots\ldots(39).$$

31. Comparing (38) and (39), and bearing in mind the formulæ of Art. 25, we are led to the system of relations:

$$Tt = P\rho \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(40),$$

$$Tu + Ut + T\,\frac{dt}{dx} = Q\rho \;\ldots\ldots\ldots\ldots\ldots(41),$$

$$Uu + T\,\frac{du}{dx} = R\rho \;\ldots\ldots\ldots\ldots\ldots\ldots(42).$$

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that is to say,

$$\mu\,\frac{d}{dx} + \frac{d\mu}{dx} + U\mu = \frac{d\lambda}{dx}\,\frac{d}{dx} - u\,\frac{d\lambda}{dx};$$

then we have, equating the coefficients of like symbolical parts,

$$\mu = \frac{d\lambda}{dx}, \quad \frac{d\mu}{dx} + U\mu = -u\,\frac{d\lambda}{dx},$$

and also the integrable equation

$$\frac{d^2\lambda}{dx^2} + (U + u)\,\frac{d\lambda}{dx} = 0,$$

for determining $\lambda$. The final result is

$$\lambda Yy = \left(\frac{d}{dx} + U\right)\left(\lambda\,\frac{d}{dx} + \lambda u - \frac{d\lambda}{dx}\right) y = 0,$$

a symbolical decomposition.

32. This system, on making

$$\rho=\frac{1}{P},\quad T=1,\quad t=1,$$

may be replaced by

$$U+u=2q\ \dots\dots\dots\dots\dots\dots\ (43),$$

$$Uu+\frac{du}{dx}=r\ \dots\dots\dots\dots\dots\dots\ (44).$$

33. From (43) we obtain

$$U=2q-u,$$

a value which, substituted for $U$ in (44), leads to

$$\frac{du}{dx}+2qu-u^2=r\ \dots\dots\dots\dots\dots\dots(6),$$

the relation of Art. 2: and the symbolical decomposition of (1) and the change of the dependent variable from $y$ to $e^{\int udx}$ have conducted us to the same result.

34. Again, if we eliminate $u$, instead of $U$, between (43) and (44), we are led to

$$\frac{dU}{dx}-2qU+U^2=2\frac{dq}{dx}-r\ \dots\dots\dots\dots(45),$$

which would arise from the cognate form of Art. 12. And, if we make $V=U-q$, then (45) becomes

$$\frac{dV}{dx}+V^2=s\ \dots\dots\dots\dots\dots\dots(5'),$$

a form discussed in Art. 1; and the connection between the relation, deducible from (43),

$$u-q=-(U-q)\dots\dots\dots\dots\dots\dots(46),$$

and (6) and (45) indicates that substantially the same results are obtained whether we eliminate $U$ or $u$. The latter elimination corresponds to the substitution of $e^{-\int udx}$ for $y$ in (1).

35. Again, on inspecting the system of Art. 32, we see, not only that $U$ and $u$ are obtainable from the quadratic

$$u^2-2qu+r=0\ \dots\dots\dots\dots\dots\dots(47),$$

when $U$, $u$, $q$ and $r$ are constant, but also that they are obtainable from (47) when $U$ the last term of the first factor is variable, provided only that $u$, the last term of the last factor, be constant. Hence (30) of Art. 20 may be considered as standing on the line which divides equations with constant, from those with variable, coefficients.

36. So, for (29), we have the decomposition

$$U = ax \pm \sqrt{(a-b)}, \quad u = ax \mp \sqrt{(a-b)};$$

and for

$$\frac{d^2y}{dx^2} + 2b\frac{dy}{dx} + (b^2 + a - a^2x^2)\,y = 0 \ldots\ldots\ldots (48),$$

we have

$$U = b + ax, \quad u = b - ax.$$

In like manner (11) affords a symbolical decomposition in which

$$U = \frac{c}{x+a} \pm \sqrt{(-b)}, \quad u = \frac{c}{x+a} \mp \sqrt{(-b)}.$$

37. Further, the system of Art. 32 manifests the solubility of

$$\frac{d^2y}{dx^2} + \left(\frac{d\lambda}{dx} + ae^{-\lambda} + b\right)\frac{dy}{dx} + abe^{-\lambda} = 0 \ldots\ldots\ldots (49),$$

and

$$\frac{d^2y}{dx^2} + \left(\frac{d\lambda}{dx} + ae^{-\lambda} + b\right)\frac{dy}{dx} + b\frac{d\lambda}{dx} = 0 \ldots\ldots\ldots (50),$$

for the former of which we see that

$$U = \frac{d\lambda}{dx} + b, \quad u = ae^{-\lambda},$$

and for the latter

$$U = \frac{d\lambda}{dx}, \quad u = ae^{-\lambda} + b.$$

38. Simple results are deducible from (49) and (50). For example, let

$$\frac{d\lambda}{dx} = \frac{mx^{m-1}}{x^m - 1}, \quad \lambda = \log(x^m - 1), \quad e^{-\lambda} = \frac{1}{x^m - 1};$$

then we see that the forms

$$\frac{d^2y}{dx^2} + \left(\frac{mx^{m-1} + a}{x^m - 1} + b\right)\frac{dy}{dx} + \frac{aby}{x^m - 1} = 0 \ldots\ldots (51),$$

and

$$\frac{d^2y}{dx^2}+\left(\frac{mx^{m-1}+a}{x^m-1}+b\right)\frac{dy}{dx}+\frac{mbx^{m-1}y}{x^m-1}=0\ldots\ldots(52),$$

are soluble. These results somewhat resemble those of Arts. 9 and 10. It may be observed that (13) of Art. 9, multiplied into $e^\lambda$, takes the form

$$\frac{d(e^\lambda u)}{dx}+ae^{-\lambda}(e^\lambda u)-e^{-\lambda}(e^\lambda u)^2=be^{-\lambda},$$

whence

$$\frac{1}{b-ae^\lambda u+(e^\lambda u)^2}\cdot\frac{d(e^\lambda u)}{dx}=e^{-\lambda},$$

which is soluble by integration.

39. The process which conducted us to the soluble form (24) may be generalized as follows: if in (3) we suppose that

$$r=2q\lambda+\lambda^2+r_2\ldots\ldots\ldots\ldots\ldots\ldots\ldots(53),$$

we find

$$\frac{du}{dx}+2q(u+\lambda)+(u+\lambda)^2-2u\lambda=-r_2,$$

or, putting $u+\lambda=u_2$,

$$\frac{du_2}{dx}+2qu_2+u_2^2-\frac{d\lambda}{dx}-2u\lambda=-r_2,$$

and

$$\frac{du_2}{dx}+2(q-\lambda)u_2+u_2^2-\frac{d\lambda}{dx}+2\lambda^2=-r_2\ldots\ldots(54).$$

Hence, if

$$-\frac{d\lambda}{dx}+2\lambda^2=0,\ \text{ or }\ \lambda=\frac{1}{2(x+C)},$$

and

$$q-\lambda=q-\frac{1}{2(x+C)}=q_2,$$

(54) will take the form

$$\frac{du_2}{dx}+2q_2u_2+u_2^2=-r_2\ldots\ldots\ldots\ldots\ldots\ldots(55),$$

which is soluble when $r_2=0$ (see Art. 17).

40. So, take

$$r_2 = \frac{q_2}{x+C_2} + \frac{1}{4(x+C_2)^2} + r_3,$$

$$u_2 + \frac{1}{2(x+C_2)} = u_3, \quad q_3 = q_2 - \frac{1}{2(x+C_2)},$$

then (55) may be put under the form

$$\frac{du_3}{dx} + 2q_3u_3 + u_3^2 = -r_3 \ldots\ldots\ldots\ldots\ldots\ldots(56),$$

which is soluble when $r_3 = 0$.

41. Hence, if

$$r = q\left(\frac{1}{x+C} + \frac{1}{x+C_2}\right) - \frac{1}{2}\,\frac{1}{(x+C)(x+C_2)} + \frac{1}{4}\left\{\frac{1}{(x+C)^2} + \frac{1}{(x+C_2)^2}\right\},$$

(3) is soluble.

42. Further, continuing the same process, and representing by

$$\Sigma\left(\frac{1}{x+C}\right)^m$$

the expression

$$\left(\frac{1}{x+C}\right)^m + \left(\frac{1}{x+C_2}\right)^m + \ldots + \left(\frac{1}{x+C_n}\right)^m,$$

we conclude that if

$$r = q\Sigma\left(\frac{1}{x+C}\right) - \frac{1}{2}\Sigma\left\{\frac{1}{(x+C)(x+C_2)}\right\} + \frac{1}{4}\Sigma\left(\frac{1}{x+C}\right)^2,$$

(3) will be soluble. We find indeed that it becomes

$$\frac{du_n}{dx} + 2q_nu_n + u_n^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots(57),$$

in which equation

$$q_n = q - \frac{1}{2}\Sigma\left(\frac{1}{x+C}\right), \quad u_n = u + \frac{1}{2}\Sigma\left(\frac{1}{x+C}\right).$$

43. Since

$$x^m\frac{d^my}{dx^m} = (kx)^m\frac{d^my}{k^mdx^m} = (kx)^m\frac{d^my}{d(kx)^m} \ldots\ldots(58),$$

if $k$ be constant, the change of the independent variable from $x$ into $kx$ does not affect expressions of the last form.

44. Consequently, if in

$$(h^2 - x^2)\frac{d^2y}{dx^2} + ax\frac{dy}{dx} + b = 0 \text{.............} (59),$$

where $h$, $a$, and $b$ are constants, we change $x$ into $\frac{x}{h}$, that equation will be transformed into

$$(1 - x^2)\frac{d^2y}{dx^2} + ax\frac{dy}{dx} + b = 0 \text{.............} (60),$$

and if the solution of (60) be $y = \phi(x)$ that of (59) will be $y = \phi(hx)$.

45. It may be well to observe that, as the algebraic equation

$$x^2 + 2qx + r = 0$$

is deprived of its second term by the substitution of $z - q$ for $x$, so (1) is deprived of its second term by the substitution of $e^{\int (u-q)dx}$ for $y$.

4, Pump Court, Temple, London,
*January* 15, 1862.

(*To be continued.*)

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# SOME REMARKS ON THE STUDY OF PURE GEOMETRY.

By *H. W. Challis*, Merton College, Oxford.

All Pure Mathematics may be divided into two sciences: Geometry, whose subject matter is the properties of pure space; and Algebra (including the Calculus), whose subject-matter is the properties of pure number. But since by regarding any magnitude as a unit, every homogeneous magnitude may be expressed as a number, it follows that to every property of number there corresponds a property of space; on which consideration what are called Analytical Geometrical Methods are founded. The Science of Number is of little importance except as regards its application to Geometry, to which latter science it is properly subordinate. So complete, however, is the investigation of Geometry by these Analytical Methods, and so comprehensive are the results obtained, that the study of Pure Geometry has been comparatively neglected since such methods came into use; while the study of the science of number has received an extraordinary stimulus, and has come to be regarded as almost the only road to the successful study of Geometry. This neglect of Pure Geometry seems to be owing, at least in part, to the method by which its elements are now generally taught. This method is defective in one way, and vicious in another. Firstly, it is unmethodical in its arrangement, and uses clumsy and inelegant expedients to arrive at its results. Secondly, it is metaphysically incorrect; that is, the means it uses, or professes to use, for communicating and rendering evident its truths, are not in accordance with psychological laws. It is true that the mind has only one means of arriving at truth, and whenever truth is apprehended, this means is used either consciously, or unconsciously. But it makes a great difference whether the student consciously uses a right method, or unconsciously does so, supposing himself to be following one that is radically wrong. If instruction were the good to be gained from mathematical studies, it would matter little which he did; but if he be seeking education, it is of paramount importance that he should consciously follow the right. To the first objection (which is comparatively trifling) the works of Euclid, and the early Geometers, lie open, as was inevitable; but they are not amenable to the second; I believe the method pursued by the early Greek Geometers to have been metaphysically correct. Indeed the doctrine to which I shall endeavour to direct attention does not differ in its practical effects from Plato's doctrine that Mathematical knowledge was apprehended by ἀνάμνησις, or recollection. It is only his explanation of certain phenomena which is supposed to be wrong. He taught that such knowledge had been acquired in a former state of existence, and having been since forgotten, was recollected when suggested to the mind by circumstances. Now it would rather appear that the presence of all such truth innately in the mind is one of the conditions (i.e. inseparable consequences) of rationality; but (I speak only from my own experience) the mental phenomena accompanying the acquirement of such truth, do bear a striking resemblance to those accompanying the recollection of some half forgotten occurrence, which is gradually recalled to the memory by circumstances.

A good instance of the inelegance above referred to is presented by the fifth proposition in Euclid. I shall not be suspected of underrating Euclid's powers if I deny his processes that elegance, which we, who inherit the fruit of his labours, might easily give them. A neater way of proving the first half of the proposition seems to consist in regarding the isosceles triangle in the figure as two triangles, taking the equal sides in a different order in each; we have thus two triangles to which the fourth proposition applies, and it follows that the angles at the base are equal. The remainder of the proposition follows from the thirteenth, when we have learnt that the lineal angle is always equal to two right angles. Thus the "happy contrivance," as Mr. J. S. Mill* calls the construction, is rendered superfluous. It was indeed a happy contrivance in Euclid's days, and shews the genius of its contriver; but it would reflect small credit on us to be content that the matter should rest as he left it. Those who lay the foundations of an illimitable study must be content with accuracy; they cannot have elegance. It is for us, who come after them, to remedy this defect.

The fundamental mistake, which is at the bottom of the second error, is the notion that the truths of Mathematics can be logically deduced from definitions and axioms. This view seems to be held by Mr. Mill,† to the reception of whose opinions respecting the inductive character of axioms it is almost, if not quite, necessary. We need go no farther than the first proposition in Euclid to see the falsity of this. By no syllogistic process can it be deduced from anything before assumed, that the circles in the figure intersect; and if they do not, the proof is futile. How then do we know that they intersect? I conceive, by intuition.

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* Logic, Vol. I. p. 245.

† Ibid, Vol. I. p. 256.

It would be out of place to enter here upon a metaphysical discussion for the purpose of shewing that axioms are apprehended intuitively; it remains to point out what consequences will follow if we accept the doctrine. It is usual to object that, admitting the intuitive character of axioms, it is inconsistent to insist on logical demonstration before we admit more abstruse geometrical propositions. The obvious reply to this objection is to admit its validity, and insist no more on logical demonstration; and anyone may see from every proposition in Euclid that it was never given, if insisted on. There is not one of his propositions that does not require an act of intuition at some stage. We should convert demonstration into a means for supplying the conditions necessary for apprehending by intuition the truth under discussion. 'Proving' a proposition would assume the form of constructing a ladder, by which limited human faculties may ascend by easy stages to the apprehension of a truth; which, had they been perfect, would have been apprehended at once; and which some with stronger faculties actually do apprehend with fewer steps. The last round of our ladder will be intuitively perceiving either that the proposition must be so, or that it cannot possibly be otherwise.

A few words remain to be said on the nature and function of problems: that is, propositions requiring something to be done. They serve two purposes: firstly, they shew that some proposed construction is possible; secondly, it often happens that by considering the construction, we accidentally perceive some new property of the figure constructed; as by considering Euc. IV. 5 we see that three points suffice to determine a circle. Such a problem as Euc I. 10., is superfluous; for nothing can render it more obvious to the student that a straight line can be bisected, than it was before: indeed, unless this, or an equivalent proposition, be tacitly assumed, the proof fails. Geometry is a science, not an art; hence it need not require to be able practically to construct every figure which it uses; it is sufficient to know that such a figure can be constructed. Our power of constructing it cannot affect its properties when constructed; and it is with these that we are concerned. Those, moreover, who appear to have thought it necessary to give every construction they used, were compelled to omit it in some cases; for instance, Euclid in his fourth proposition does not shew us how to move figures and superinduce them upon one another.

Subjoined are some definitions upon which might be erected a system of elementary Geometry. Mere technical definitions have been excluded, and the consideration confined to fundamental conceptions. It may perhaps appear inconsistent to omit 'surface'; but it is omitted for the present from motives of practical convenience. As the following are not logical definitions, but contain implied and expressed axioms, I have preferred to give them a different name.

## PRELIMINARY CONSIDERATIONS.

I. A 'point' is the limit to which all magnitudes, existing in space, tend, as they are diminished more and more; consequently no magnitude can be conceived so small as a point, and thus no parts of a point can be discerned. As a point exists in space, it has position. Position being the only attribute of a point, we may use indifferently the terms 'point' and 'position of a point.'

II. Since a point has position, it may be conceived to move from one position to another; it will thus successively occupy and relinquish a series of positions; the aggregate of these forms our conception of a 'line.' This will explain what is meant by the term 'generate,' when we say that 'a line is generated by the motion of a point from one position to another.'

III. "A straight line," says Euclid, "is that (line) which lies evenly (ἐξ ἴσου) between its extreme points." This requires some explanation. If a point be conceived to move from one position to another consecutive to the first, these two points (being distinguishable in conception) will have a determinate relation to one another in respect of position: this relation is called 'direction.' We interpret 'lying evenly' to mean 'having the same direction between every two consecutive points.' Thus if a straight line be drawn between two fixed points, we can conceive its 'production,' or generation beyond those points; for the generating point will continue to move so as always to preserve the same direction between two consecutive points. Since the generating point can move only in one direction, only one straight line can be drawn between two points; that is, 'two straight lines cannot enclose space.'

Some have proposed to define a straight line as the 'shortest that can be drawn between two fixed points.' But Kant has pointed out that this proposition is a synthetical judgment, inasmuch as we go out of our original conception of 'straight,' which is of quality only, to add something of quantity. If we were to accept this definition, we could not analytically (i.e., without going out of our original conception) conceive the production of a straight line.

IV. A 'plane' is generated by the motion of a straight line, which always passes through a fixed point, and has a point in common with a fixed straight line not passing through the fixed point. The moving line is called the 'generator,' the fixed line the 'director.'

V. If in a plane several straight lines be conceived passing through one point, we perceive that any two of them differ in respect of their position with regard to a third. This difference is commonly expressed by saying that the former lines are differently 'inclined' to the latter; and 'the inclination' to one another of two straight lines* which meet together, is called an angle. Though it is sometimes convenient to use this term in a loose sense, to signify the point where the two lines intersect (i.e., the angular point,) it must be remembered that 'angle' connotes only a fact respecting the position of two straight lines with regard to one another; which lines are said to contain the angle.

We have seen that angles may differ; the difference is commonly expressed by saying that one is 'greater' or 'smaller' than another; but such terms can only be applied to an abstraction, like 'angle,' by analogy. To find for them a determinate meaning, let us suppose that we are considering the relation to one another of two given angles: let one of the lines containing one of them be applied to one of those containing the other, the angular points coinciding, and the other lines containing the angles being both on the same side of the common line. Now if these other lines coincide, we term the angles 'equal'; if not one of them must be included between the other and the common line; the angle contained by the included line and the common line is said to be 'smaller' than the other.†

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* The term 'angle,' as here used, applies only to the 'rectilinear angle' of Euclid; but it will be seen hereafter that there is no intelligible conception of a curvilinear angle, except the rectilinear angle contained at the point of intersection of the curves, by portions of the curves so small as to be undistinguishable in conception from straight lines.

† 'Inclination' is the relation to one another of two 'directions.' A straight line may be said to have a direction, as the direction between any two of its consecutive points is the same: in a curved line the direction between any two consecutive points is not generally the same; that is, 'the direction at any point is variable.' If we define 'parallel' straight lines as 'those which have the same direction,' Euc. I. 29 follows directly.

# THEOREM IN MAXIMA AND MINIMA.

By *H. J. Purkiss,* Scholar of Trinity College, Cambridge.

I. I HAVE met with numerous examples of *maxima* and *minima*, which have appeared to me to be particular cases of a general *theorem.* This theorem we may enunciate as follows:

*Theorem.* If $u$ be a symmetrical function of $n$ variables, $x_1, x_2, x_3, \ldots x_n$, which are connected by not more than $n-1$ symmetrical relations; and if $x_1, x_2, x_3, \ldots x_n$ can simultaneously have the same value, then when they all have this value, $u$ is a *maximum* or *minimum.*

II. First, if $u$ be a symmetrical function of $x$ and $y$, which are connected by a symmetrical relation, then when $x=y$, $u$ is a maximum or minimum.

Let $$u=\phi(x, y),$$
where $$\psi(x, y)=0.$$
We here suppose that $\psi(x, y)$ does not contain $x-y$ as a factor; for if so, there would be an infinite number of equal values of $x$ and $y$, and we should not know which of these to select.

Suppose when $x=y$ each $=a$.

Then if we make
$$x=a+\delta a,$$
we have
$$y=a+\delta' a,$$
where $\delta a$, $\delta' a$ are very small *finite* quantities which need not be taken equal, for if so, we should have $x=y$ for all values of each lying between $a$ and $a+\delta a$, and therefore $\psi(x, y)$ would be divisible by $x-y$, which we suppose not to be the case. Hence we may assume $\delta a$, $\delta' a$ to be unequal.

Therefore if $x=a+\delta a,\quad y=a+\delta' a,$
we have
$$u=\phi(a+\delta a,\ a+\delta' a)$$
$$=\phi(a+\delta' a,\ a+\delta a),$$
since the function is symmetrical.

Now evidently when $x=a+\delta' a$, $y$ can be equal to $a+\delta a$. Hence $u$ is constant when $x$ has a small variation in the neighbourhood of $a$. Therefore $u$ is a maximum or minimum when $x=y=a$.

III. The case in which $\delta a$, $\delta' a$ are ultimately equal may be illustrated geometrically. Let $x$, $y$ (fig. 29) be the Cartesian coordinates of a point. Then $\psi(x, y)=0$ represents a curve symmetrical with respect to the bisector of $yOx$; and where the curve cuts this line it will cut it at right angles, as at $V$, or at a multiple point, as at $P$, $T$.

The case we wish to consider is when $\frac{dy}{dx}=1$, then we have a cusp as at $P$. Let the abscissa of $P=a$, and let $QSRNM$ be an ordinate at distance $PN=\delta a$ from $P$, cutting the curve in $Q$, $R$, and the straight line $OP$ in $S$. Then when $\delta x=PN$, $\delta y=NR$ or $NQ$, neither of which is equal to $NS$ or $\delta a$. Therefore until we proceed to the limit, $\delta' a$ is not equal to $\delta a$.

IV. Suppose now that we have

$$u=F(x_1, x_2, \dots x_n),$$

and

$$f(x_1, x_2, \dots x_n)=0,$$

where the two functions $F$, $f$ are symmetrical.

Put $x_1=x_2=\dots=x_n=a$ suppose,

and let us vary any two of the variables, *e.g.* $x_1$, $x_2$. Then

$$u=F(x_1, x_2, \dots a),$$

$$f(x_1, x_2, \dots a)=0.$$

Hence by (II.) $u$ is a maximum or minimum, as far as a variation of $x_1$ and $x_2$ is concerned, when $x_1=x_2=a$. So for a variation of any two of the variables. Hence $u$ is a maximum or minimum for variations of all the variables. Hence the theorem is proved in the case when there is only one symmetrical relation connecting the variables, which is the case with which we most frequently meet.

V. Next, let us suppose that there are *two* symmetrical relations between the variables, and that they *can* simultaneously have the same value.

If we eliminate one of the variables between the two relations, we shall have the remaining variables connected by a single symmetrical relation, and $u$ will be a symmetrical function of these same variables. Hence this case is reduced to the preceding: and by induction we see that the general theorem is true, where the variables are connected by not more than $n-1$ relations, which admit of the variables having the same value simultaneously.

### EXAMPLES.

(1) Find the minimum value of

$$u=\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\ldots+\frac{1}{x_n},$$

where

$$x_1^p+x_2^p+\ldots+x_n^p=x_1^q+x_2^q+\ldots+x_n^q=x_1^r+x_2^r+\ldots+x_n^r.$$

If $x_1$ can be $=x_2=x_3=\ldots=x_n$,

$$x_1^p=x_1^q=x_1^r;$$

therefore $$x_1=1,$$

consequently the solution is

$$x_1=x_2=\ldots=x_n=1,$$

and $$u=n.$$

(2) Find the minimum value of $u$ from the equation

$$u=x^2+y^2+z^2+\ldots,$$

the variables being connected by the equation

$$ax+by+cz+\ldots=k.$$

(Todhunter's *Diff. Calc.*, Chap. XVI., Ex. 6.)

We have $$u=a^2\left(\frac{x}{a}\right)^2+b^2\left(\frac{y}{b}\right)^2+\ldots,$$

and $$a^2\left(\frac{x}{a}\right)+b^2\left(\frac{y}{b}\right)+\ldots=k;$$

therefore $$\frac{x}{a}=\frac{y}{b}=\ldots=\frac{x^2+y^2+\ldots}{ax+by+\ldots}=\frac{ax+by+\ldots}{a^2+b^2+\ldots};$$

therefore $$u=\frac{k^2}{a^2+b^2+c^2+\ldots}.$$

(3) A given volume $V$ of metal is to be formed into a rectangular vessel; the sides of the vessel are to be of a given thickness $a$, and there is to be no lid. Determine the shape of the vessel so that it may have a maximum capacity. (Todhunter's *Diff. Calc.*, Chap. XVI., Ex. 11.)

Apply together two such vessels equal in every respect, so as to form a closed rectangular vessel. The depth of this vessel will be double that of the former; and the capacity of one will be a maximum when that of the other is also.

Let $x$, $y$, $z$ be the external length, breadth, and thickness; $x$, $y$, $z$ will clearly be connected by a symmetrical relation, and the capacity will be a symmetrical function of them. Therefore $x = y = z$. Or, going back to the former vessel, $x = y = 2z$.

(4) A pencil of rays is refracted through a prism in a principal plane, to find when the deviation of the axis is a minimum.

Using the notation of Parkinson's *Optics*, Art. 93, we have

$$D = \psi + \phi - i,$$

$$\sin^{-1}\frac{\sin\phi}{\mu} + \sin^{-1}\frac{\sin\psi}{\mu} = i;$$

therefore $$\phi = \psi.$$

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## GEOMETRICAL NOTES.

By *J. Mc Dowell, B.A., F.R.A.S.*, Pembroke College, Cambridge.

1. PROVE geometrically that in any plane triangle

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

Let the perpendiculars from the angles to the opposite sides meet in $P$ (fig. 30).

Then $$\cot C = \cot BPD = \frac{DP}{BD},$$

$$\cot B = \cot CPD = \frac{DP}{CD};$$

therefore $$\cot B \cot C = \frac{DP^2}{BD.CD} = \frac{DP^2}{AD.DP} = \frac{DP}{AD} = \frac{\Delta BPC}{\Delta ABC}.$$

Similarly, $$\cot C \cot A = \frac{\Delta CPA}{\Delta ABC},$$

and $$\cot A \cot B = \frac{\Delta APB}{\Delta ABC};$$

therefore

$$\cot B \cot C + \cot C \cot A + \cot A \cot B = \frac{\Delta BPC + CPA + APB}{\Delta ABC} = 1;$$

therefore $\tan A + \tan B + \tan C = \tan A \ \tan B \ \tan C$. Q.E.D.

2. On opposite sides of any straight line subtending a right angle, equal segments of circles are described; shew that the four points in which the circles, to which the segments belong, again meet the sides of the right angle, are the angular points of a parallelogram.

Let the straight lines $CF$ and $ED$ intersect at right angles in $O$ (fig. 31), and let $AB$ subtend the right angle $EOF$, and let the equal segments $AEFB$ and $ACDB$ be described on $AB$.

Draw the other straight lines as in the figure.

The angle $BFE$ of quadrilateral

$$ABFE = \text{external angle } DAB,$$
$$= DCB \text{ in same segment};$$

therefore $CD$ and $EF$ are *always* parallel (whatever angle $O$ may be).

Also $$\angle AFB = ADB \text{ (hyp.)};$$

therefore since angles at $O$ are right angles,

$$\angle OAF = CBD;$$

therefore since the two circles are equal, the chords $EF$ and $CD$ which respectively subtend supplemental angles at $A$ and $B$ are equal; but $EF$ and $CD$ have also been proved to be parallel; therefore $C$, $E$, $F$, $D$, are the angular points of a parallelogram. Q.E.D.

3. If a circle with centre $O$ meet *any* hyperbola in the points $A$, $B$, $C$, $D$, (fig. 32), and if the common chords $AB$, $CD$ be produced to meet the asymptotes in $A'$, $B'$, $C'$, $D'$; these points will also lie in a circle with centre $O$.

By a known property of the hyperbola,

$$A'A.AB' = C'C.CD',$$

and

$$A'A = BB',$$

$$C'C = DD'.$$

Since triangles $AOB$, $OCD$, are isosceles as also obviously $A'OB'$, $C'OD'$,

we have

$$\begin{aligned} A'O^2 &= OA^2 + A'A.AB' \\ &= OC^2 + C'C.CD' \\ &= OC'^2; \end{aligned}$$

therefore $A'O = C'O = OB' = OD'$. Q.E.D.

Cor. If on the opposite side of the chord $AB$ of the hyperbola there be described a segment of a circle equal to $ACDB$ with centre $O'$, it is very easily seen that if this circle with centre $O'$ meet the hyperbola again in the points $L$, $M$, and if $LM$ produced meet the asymptotes in $L'$, $M'$, then $L'$, $M'$, $A'$, $B'$, will lie in a circle with centre $O'$ and radius equal to that of circle through $A'$, $B'$, $C'$, $D'$.

4. "On opposite sides of any chord of a rectangular, hyperbola are described equal segments of circles; shew that the four points, in which the circles, to which these segments belong, again meet the hyperbola, are the angular points of a parallelogram."

This theorem set (by the Rev. J. Wolstenholme, I believe) at the Mathematical Tripos Examination, 1862, follows very simply from 2 and 3 given above; 2 is only a particular case of it, but 3 is, so far as I am aware, a new theorem.

Let $AB$ (fig. 33) be any chord of a rectangular hyperbola and let it meet the asymptotes in $A'$, $B'$. Also let any equal circles on opposite sides of $AB$ meet the curve again in the points $C$, $D$, $L$, $M$, and let $CD$, $LM$ meet the asymptotes in $C'$, $D'$; $L'$, $M'$.

Therefore, by (3) and Cor. equal circles on opposite sides of $A'$, $B'$ meet the asymptotes in $C'$, $D'$; $L'$, $M'$.

Therefore since the asymptotes intersect at right angles, $L'$, $M'$, $C'$, $D'$ are the angular points of a parallelogram by (2).

Therefore since $L'M' = C'D'$, we have by a known property of the hyperbola $CD = LM$; therefore $C$, $D$, $L$, $M$ are the angular points of a parallelogram.

Q. E. D.

Let $M$, $O$, $O_1$, $O_2$, $O_3$ (fig. 34) be the centres of the circumscribed, inscribed, and escribed circles of the triangle $ABC$; $R$, $r$, $r_1$, $r_2$, $r_3$ the radii of these circles.

By known properties (which are very easily proved geometrically) the straight line $O_3AO_2$ is bisected by the circumscribing circle in $F$, and the diameter $FE$ bisects $BC$ and $GH$ each in $D$, and $OO_1$ is bisected by the circle in $E$.

Since $GO_3$, $DF$, and $HO_2$ are parallel and $GD = DH$; therefore

$$r_2 + r_3 = 2FD,$$

and because $O_1E = EO$ and $O_1L$, $ED$ and $OK$ are parallel; therefore

$$r_1 - r = 2DE.$$

Hence $$r_1 + r_2 + r_3 - r = 2FD + 2DE = 2FE = 4R \ldots\ldots (\alpha).$$

Let $p$, $p'$, $p''$ denote the perpendiculars from centre of circumscribed circle to sides, $q$, $q'$, $q''$ the parts of these perpendiculars between the sides and the circumscribed circle, so that $q = ED$ and $p = DM$.

Therefore, as above,

$$r_1 - r = 2DE = 2q;$$

therefore $$2q + 2q' + 2q'' = r_1 + r_2 + r_3 - 3r$$
$$= 4R - 2r \text{ by } (\alpha);$$

therefore $$q + q' + q'' = 2R - r \ldots\ldots\ldots\ldots\ldots\ldots (\beta),$$

but obviously $$\Sigma(p) + \Sigma(q) = 3R;$$

therefore $p + p' + p''$ or $\Sigma(p) = 3R - (2R - r) = R + r \ldots (\gamma)$.

The Perse Grammar School,
*January* 27, 1862.

N.B.—For a different proof of ($\gamma$) see Catalan's *Geometrical Exercises*, third edition, 1858, p. 89; or *The Quarterly Journal of Mathematics*, for March, 1862, p. 166.

# HINTS ON THE THEORY OF GROUPS.

By the *Rev. T. P. Kirkman, M.A., F.R.S.*, Honorary Member of the Literary and Philosophical Societies of Manchester and Liverpool.

(*Continued from page* 68.)

10. *Construction of m-valued functions.* The method of constructing a many-valued function upon a given group, which has been in vogue for the last twenty or thirty years at Paris, where this subject has been considered and perhaps lectured on since the earlier days of Cauchy, is thus laid down in one of the very last French contributions to this theory. The author is shewing how functions of $r$ values may be constructed on groups of the order $\frac{\underline{|n}}{r}$.

"On se donne une expression dissymétrique (*sic*) par rapport aux $n$ lettres $a$, $b$, $c$ ...; celle-ci par exemple, $\alpha a+\beta b+\gamma c+\ldots$: puis on forme une fonction symétrique des valeurs que prend cette expression en y effectuant toutes les substitutions du système conjugué; on prendra, par exemple, le produit

$$(\alpha a+\beta b+\ldots)(\ldots)(\ldots).$$

Cette fonction sera invariable par toutes les substitutions du système, qui ne font que permuter les facteurs les uns dans les autres; car si $P$, $Q$, $R$ ... sont les substitutions du système conjugué, et si l'on désigne par $F_x$ ce que devient la fonction lorsqu' on y a effectué la substitution $x$, le produit sera représenté par $F_1 F_P F_Q \ldots$; après la substitution $P$ il sera devenu $F_P F_{P^2} F_{QP} \ldots$. Mais les substitutions $P_1 P_1^2 QP \ldots$ sont évidemment distinctes, et appartiennent toutes au système conjugué: ce sont les substitutions $1PQ \ldots$ écrites dans un autre ordre. D'un autre côté toute autre substitution, altérant essentiellement les facteurs, altèrera le produit." (*Thèses présentées à la Faculté des Sciences de Paris, etc.*, p. 10, Paris, Mallet-Bachelier, 1860. Vide also a reprint of the same memoir in the *Journal de l'école Polytechnique*, Vol. XXII., 1861.)

By the letters $abc\ldots$ the author means the elements of unity. Let us try the method on the simple group, (système conjugué),

$$\begin{array}{l} abcd,\ 1, \\ bcda,\ P, \\ cdab,\ Q, \\ dabc,\ R, \end{array}$$

which is the group $G'$ (Art. 2). We may take any symmetric function of $abcd$. Take, as instructed, $ac+bd=F_1$. The product which we are directed to form is

$$(ac+bd)\,(bd+ca)\,(ca+db)\,(db+ac)=F_1F_PF_QF_R.$$

It is perfectly true that this product is unaltered by any of the four substitutions,

$$1,\ \frac{bcda}{abcd},\ \frac{cdab}{abcd},\ \frac{dabc}{abcd}.$$

Let us try any one of the four substitutions $adcb$, $dcba$, $cbad$, $badc$,

that is $$\frac{adcb}{abcd},\ \frac{dcba}{abcd},\ \frac{cbad}{abcd},\ \frac{badc}{abcd},$$

all different from $1$, $P$, $Q$, and $R$. Every one of them, by the concluding words above quoted, ought to alter the product. The first gives us the product

$$(ac+db)\,(db+ca)\,(ca+bd)\,(bd+ac);$$

the second gives us

$$(db+ca)\,(ca+bd)\,(bd+ac)\,(ac+db);$$

the third gives us

$$(ca+bd)\,(bd+ac)\,(ac+db)\,(db+ca);$$

the fourth gives us

$$(bd+ac)\,(ac+db)\,(db+ca)\,(ca+bd).$$

So far is the product $F_1F_PF_QF_R$ from being altered and taking one or more different values by these four substitutions, it is not altered by any of them. This French rule will be found equally misleading in the treatment of innumerable groups, both simple and complex. It is hardly fair to make the author of the *Thèses* responsible for this error, as he is not at all giving his own results in this first chapter; he is merely stating by way of brief introduction what has long been established at Paris. His words in the preface are: "Dans le chapitre 1$^{\text{er}}$, je reprends les fondements du sujet. J'établis le théorème de Lagrange et quelques autres principes genéraux: la presque totalité de ce chapitre est emprunteé à M. Cauchy. J'ai cru faire une chose utile en résumant ainsi ces travaux importants et fort peu connus."

The author has proved himself in the following part of his work to be perfectly competent to cope with this difficult subject; and has repeated above, about the construction of functions, an oversight which will be most likely found in all previous French writers, namely, the oversight of this principle;—that in order to prove that a function of $n$ letters has $r$ values, it is not enough to show that it is *invariable* by the $\frac{\underline{|n}}{r}$ substitutions of a certain group; but that it is required to be demonstrated, also that it is *variable* by every other substitution.

11. THEOREM VII. *Let $G$ be any model group of the $K^{\text{th}}$ order made with $n$ elements, and let*

$$P_1 = x_1^\alpha . x_2^\beta . x_3^\gamma \dots x_n^\nu$$

*be the product of any $n$ different powers ($\geqq 0$) of the $n$ variables $x_1 x_2 \dots x_n$.*

Let $$\psi = P_1 + P_2 + P_3 + \dots + P_K$$

*be the sum of the $K$ terms obtained by performing on the sub-indices of $P_1$ the substitutions of $G$. The function $\psi$ has $\frac{\underline{|n}}{K}$ values by the permutations of the $n$ variables.*

Any permutation of the variables in $\psi$ is effected by performing on all its terms a certain substitution

$$\phi = \frac{x_a x_b x_c \dots x_r}{x_1 x_2 x_3 \dots x_n},$$

where the sub-indices above are those below in a different order. And this substitution is performed by the operation of

$$\phi = \frac{abc\ldots r}{123\ldots n}$$

on the sub-indices, as in Art. 3. This result may be written as $\phi\psi$, for $\psi$ is here the subject of operation by the substitution $\phi$, which operates on every term of $\psi$.

If $\phi$ be any of the substitutions of $G$, it is plain that $\phi\psi$ and $\psi$ will differ only in the order of the same terms $P_1P_2\ldots P_K$, and that there will be no change of value in $\psi$. It will be observed that in this operation with $\phi$ the exponents of $x_1x_2\ldots$ are nowhere disturbed, and that

$$P_a = x_e{}^{\alpha}x_f{}^{\beta}x_g{}^{\gamma}\ldots, \text{ and } P_b = x_p{}^{\alpha}x_q{}^{\beta}x_r{}^{\gamma}\ldots$$

merely change their places for others, if $\phi$ be a substitution of $G$ which turns *efg*... into some other arrangement of $G$, and *pqr*... into some other; as $\phi G$ is only $G$ in a different order.

If $\phi_1$ is not a substitution of $G$, $\phi_1 G$ is a derivate of $G$, which, by Theorem IV., contains no substitution of $G$. Hence every term in $\phi_1\psi$ will have a value different from that of every term in $\psi$, and $\phi_1\psi$ is a value of $\psi$ which is unaltered (Theorem V.) by any substitution in $\phi_1 G_1$. Our hypothesis that no two exponents $\alpha\beta\ldots$ are alike, renders it impossible that a term of $\phi_1\psi$ can be equal to one of $\psi$. If the exponents were not all unlike, it might easily happen that $\phi_1\psi$ should have the same series of terms with $\psi$; for the disturbance of sub-indices under the same exponents might make no change of algebraic value in $P_1P_2P_3\ldots$.

The value $\phi_1\psi$ is constructed by writing $x^{\alpha}x^{\beta}x^{\gamma}\ldots$ over all the substitutions of the derivate $\phi_1 G$.

Let now $\phi_{11}$ be any substitution not found either in $G$ or in $\phi_1 G$. We construct a new value $\phi_{11}\psi$, by operating with $\phi_{11}$ on all the sub-indices of $\psi$, or, what is the same thing, by writing $x^{\alpha}x^{\beta}x^{\gamma}\ldots$ over all the arrangements of the derivate $\phi_{11} G$.

In the same way it is proved that $\frac{\underline{|n}}{K} - 1$ values of $\psi$

can be formed by writing (Theorem VI.) the same $x^\alpha x^\beta x^\gamma\ldots$ over all the terms of all the derivates of $G$.

The simplest way to form the values of a function given by a group $G$ is to write out $G$ and its derivates, and then to add the system chosen of exponents. And we can write

$$1^\alpha 2^\beta 3^\gamma \ldots \text{ for } x^\alpha{}_1 x^\beta{}_2 x^\gamma{}_3 \ldots.$$

Take for example the group $H$, and let the system of exponents be $(\alpha\beta\gamma\delta = 3210)$.

The group with its two derivates is

| | | |
|---|---|---|
| 1234 | 1342 | 1423 |
| 2341 | 3421 | 4231 |
| 3412 | 4213 | 2314 |
| 4123 | 2134 | 3142 |
| 1432 | 1243 | 1324 |
| 4321 | 2431 | 3241 |
| 3214 | 4312 | 2413 |
| 2143 | 3124 | 4132; |

whence we have the three-valued function

$$\psi = 1^3 2^2 3 + 2^3 3^2 4 + 3^3 4^2 1 + 4^3 1^2 2 + 1^3 4^2 3 + 4^3 3^2 2 + 3^3 2^2 1 + 2^3 1^2 4,$$

whose other two values are

$$\psi_1 = 1^3 3^2 4 + 3^3 4^2 2 + 4^3 2^2 1 + 2^3 1^2 3 + 1^3 2^2 4 + 2^3 4^2 3 + 4^3 3^2 1 + 3^3 1^2 2,$$

$$\psi_2 = 1^3 4^2 2 + 4^3 2^2 3 + 2^3 3^2 1 + 3^3 1^2 4 + 1^3 3^2 2 + 3^3 2^2 4 + 2^3 4^2 1 + 4^3 1^2 3.$$

Any substitution effected on any of these values will produce one of the three values.

For example, if we operate on the last value with $\frac{1243}{1234}$, we get

$$1^3 3^2 2 + 3^3 2^2 4 + 2^3 4^2 1 + 4^3 1^2 3 + 1^3 4^2 2 + 4^3 2^2 3 + 2^3 3^2 1 + 3^3 1^2 4,$$

the same value. If we operate with $\frac{1324}{1234}$, we obtain

$$1^3 4^2 3 + 4^3 3^2 2 + 3^3 2^2 1 + 2^3 1^2 4 + 1^3 2^2 3 + 2^3 3^2 4 + 3^3 4^2 1 + 4^3 1^2 2,$$

the first value.

In the same manner can be constructed in $H'$ and $H''$ two other three-valued functions $\chi$ and $\omega$ of the sixth degree, such that the nine values of $\psi\chi\omega$ shall all be different. And these are all the three-valued functions possible of the sixth degree, which are homogeneous and have the exponents 321, and have not more than eight terms.

We can form functions on the same groups of a lower degree. We may take the system of exponents ($\alpha\beta\gamma\delta = 2100$) and construct on $H$ above written the function

$$\sigma = 1^2 2 + 2^2 3 + 3^2 4 + 4^2 1 + 1^2 4 + 4^2 3 + 3^2 2 + 2^2 1,$$

which has three values.

If we apply the same system of exponents to the group $H'$, we obtain

$$\sigma_1 = 1^2 2 + 3^2 1 + 4^2 3 + 2^2 4 + 1^2 3 + 2^2 1 + 4^2 2 + 3^2 4,$$

a three-valued function also; but the student can easily shew that this is merely a value of $\sigma$, and that the two groups $H$ and $H'$ do not give by this system of exponents, as above, *two equivalent functions*, of which neither is a value of the other, but *two values of the same function.*

The group $H''$ however gives a different three-valued function of the third degree, which is $1^2 2 + 4^2 3 + 2^2 1 + 3^2 4$, of four terms only.

We may even, in this case, employ the exponents ($\alpha\beta\gamma\delta = 1100$). The two groups $H$ and $H'$ give two values of the same three-valued function

$$\zeta = 12 + 23 + 34 + 41,$$

while the group $H''$ gives the simpler three-valued function

$$\theta = 12 + 43.$$

In the same way we can, by any of the groups of the fourth order, obtain a distinct six-valued function of four letters; but none of these will be of a degree so low as the second. On the group $I$, we can form two-valued functions.

The theory of these functions cannot be fully opened, until the student has learned something of the nature and relations of *equivalent groups*. This, along with the elements of the *theory of substitutions* may be the subject of a future communication.

Croft Rectory, Warrington,
*Sep.* 25, 1861.

# QUADRILINEAR COORDINATES.

By *William Allen Whitworth, B.A.*, Scholar of St. John's College, Cambridge, and First Mathematical Master in Rossall School.

In using trilinear coordinates symmetry is gained at the expense of that simplicity which attends the ordinary Cartesian notation with its two variables only. Considering the great advantages which he has found to result in many cases from the introduction of the third variable, the reader will scarcely be unprepared to find that even a fourth may sometimes be advantageously employed, especially in the investigation of such theorems as involve four straight lines symmetrically.

It is our purpose in the following paper to exhibit and exemplify the use of such four coordinates.

1. Let $\alpha'$, $\beta'$, $\gamma'$ be trilinear coordinates of a point $P$, and let $\delta'$ be its distance from the straight line whose equation is

$$l\alpha' + m\beta' + n\gamma' = 0;$$

then, with the notation of a former paper, (*suprà*, p. 51)

$$l\alpha' + m\beta' + n\gamma' = \{l, m, n\}\, \delta',$$

or

$$l\alpha' + m\beta' + n\gamma' + r\delta' = 0 \quad\text{.................(1).}$$

Also, if $a$, $b$, $c$, $d$ be the sides of the quadrilateral formed by the three lines of reference and the other line, we have

$$a\alpha' + b\beta' + c\gamma' + d\delta' = \text{twice the area of the quadrilateral,}$$
$$= S, \text{ (suppose)} \quad\text{.......................(2).}$$

Thus if $\alpha'$, $\beta'$, $\gamma'$, $\delta'$ be the perpendiculars from any point on the four sides of a quadrilateral, they are connected by the relations

$$l\alpha' + m\beta' + n\gamma' + r\delta' = 0,$$
$$a\alpha' + b\beta' + c\gamma' + d\delta' = S,$$

where $l$, $m$, $n$, $r$, $a$, $b$, $c$, $d$, $S$ are constant functions of the sides and angles of the quadrilateral.

The perpendiculars $\alpha'$, $\beta'$, $\gamma'$, $\delta'$ may be used as the quadrilinear coordinates of the point $P$.

2. Let $l\alpha' = \alpha$, $m\beta' = \beta$, $n\gamma' = \gamma$, $r\delta' = \delta$: then $\alpha$, $\beta$, $\gamma$, $\delta$ will determine, as well as $\alpha'$, $\beta'$, $\gamma'$, $\delta'$, the position of the point $P$, they may therefore be used, instead of the perpendiculars, as the coordinates of the point, and it will generally be more convenient to use them.

Let $\frac{a}{lS} = A$, $\frac{b}{mS} = B$, $\frac{c}{nS} = C$, $\frac{d}{rS} = D$, then the two equations, which the coordinates $\alpha$, $\beta$, $\gamma$, $\delta$ always satisfy, become

$$\alpha + \beta + \gamma + \delta = 0 \ldots\ldots\ldots\ldots\ldots\ldots(3),$$

$$A\alpha + B\beta + C\gamma + D\delta = 1 \ldots\ldots\ldots\ldots\ldots(4),$$

3. The equation (3) will play a very important *rôle* in the manipulation of equations in quadrilinear coordinates. We will call the reader's attention at once to some of its principal effects.

I. It enables us, at any time, to reduce our quadrilinear equations to trilinear, by eliminating one of the variables ($\delta$ suppose) by the substitution

$$\delta = -(\alpha + \beta + \gamma).$$

And since this transformation cannot affect the degree of an equation, it follows that, as in trilinear coordinates, an equation of the first degree represents a straight line, of the second a conic section, &c.

II. It gives us the power of eliminating, from any equation, terms of any particular argument that we may wish to get rid of. For instance we may without loss of generality eliminate, from the general equation of the second degree, the terms whose arguments are $\alpha^2$, $\beta^2$, $\gamma^2$, $\delta^2$ by substituting

$$\alpha^2 = -(\alpha\beta + \alpha\gamma + \alpha\delta),$$
$$\beta^2 = -(\beta\gamma + \beta\delta + \beta\alpha),$$
$$\gamma^2 = -(\gamma\delta + \gamma\alpha + \gamma\beta),$$
$$\delta^2 = -(\delta\alpha + \delta\beta + \delta\gamma).$$

Hence there is no objection to our assuming the general equation of the second degree of the form

$$\lambda\alpha\beta + \lambda'\gamma\delta + \mu\alpha\gamma + \mu'\beta\delta + \nu\alpha\delta + \nu'\beta\gamma = 0,$$

or we may omit some of these terms and retain some of the others.

III. It must be observed that the same locus can be represented by an indefinite number of different equations, formed from one another by substitutions of the equation (3). Hence we are not at liberty to infer that because two different equations,

$$l\alpha + m\beta + n\gamma + r\delta = 0,$$

and $$l'\alpha + m'\beta + n'\gamma + r'\delta = 0,$$

suppose, represent the same straight line, the coefficients of like terms in the two are proportional: we may infer no more than that the two equations, together with

$$\alpha + \beta + \gamma + \delta = 0,$$

form a system of only two independent equations.

Thus too, the equation to the straight line at infinity is not restricted to the form

$$A\alpha + B\beta + C\gamma + D\delta = 0,$$

but, more generally, that straight line is represented by any equation of the form

$$(A + \kappa)\,\alpha + (B + \kappa)\,\beta + (C + \kappa)\,\gamma + (D + \kappa)\,\delta = 0,$$

$\kappa$ being any constant.

4. Let $ABCD$ (fig. 35) be the quadrilateral of reference and let $AB$ and $DC$ meet in $E$, and $AD$ and $BC$ in $F$. Draw the diagonals $AC$, $BD$, $EF$: and let $AC$, $BD$ meet in $O$; $AC$, $EF$ in $Q$; $AC$, $BD$ in $R$.

Let $\alpha = 0$, $\beta = 0$, $\gamma = 0$, $\delta = 0$ be the equations to $AB$, $BC$, $CD$, $DA$ respectively: then, by equation (3) of Art. 2, the equations

$$\alpha + \beta = 0 \text{ and } \gamma + \delta = 0 \text{.................(1)},$$

must be equivalent, and therefore represent the same straight line. But the locus of the first evidently passes through $B$, and that of the second through $D$, hence either of them represents the diagonal $BD$.

Similarly, either of the two equations

$$\beta+\gamma=0 \text{ and } \delta+\alpha=0 \quad\ldots\ldots\ldots\ldots\ldots(2),$$

represents the diagonal $AC$, and either of the two

$$\alpha+\gamma=0 \text{ and } \beta+\delta=0 \quad\ldots\ldots\ldots\ldots\ldots(3),$$

the diagonal $EF$.

It follows that

the equation $\alpha-\gamma=0$ represents the line $OE$,
.............. $\beta-\delta=0$ ...................... $OF$,
.............. $\alpha-\delta=0$ ...................... $RA$,
.............. $\beta-\gamma=0$ ...................... $RC$,
.............. $\alpha-\beta=0$ ...................... $QB$,
.............. $\gamma-\delta=0$ ...................... $QD$,

5. From the forms of the equations in the last article we see that the pencils

$$\{A.BCDR\},\ \{B.ADCQ\},\ \&c.,$$

and therefore the pencils

$$\{Q.BCDR\},\ \{R.ADCQ\},\ \&c.,$$

and also the pencils at $E$ and $F$ are *harmonic*:—well known properties of the quadrilateral.

6. Since a conic can only be made to satisfy five conditions, it follows that the general equations of the conic described about or inscribed in the quadrilateral of reference, will each involve only one independent undetermined constant.

The general equation of a conic circumscribing the quadrilateral $ABCD$ will evidently be

$$\alpha\gamma-\kappa\beta\delta=0.$$

Similarly a conic passing through $A$, $C$, $E$, $F$ has its equation of the form

$$\alpha\beta-\kappa\gamma\delta=0,$$

and a conic through $B$, $E$, $F$, $D$, of the form

$$\alpha\delta-\kappa\beta\gamma=0.$$

7. *To find the general equation of an inscribed conic, free from the terms whose arguments are* $\alpha^2$, $\beta^2$, $\gamma^2$, $\delta^2$.

We know that the equation to any conic touching the three sides $a$, $b$, $c$, is

$$l^2\alpha^2+m^2\beta^2+n^2\gamma^2-2mn\beta\gamma-2nl\gamma\alpha-2lm\alpha\beta=0\ldots(1),$$

and the condition that the line represented by $\delta=0$ or $\alpha+\beta+\gamma=0$ should be a tangent, is

$$l+m+n=0\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

we have also

$$\alpha+\beta+\gamma+\delta=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Eliminating $l$ from (1) and (2), we get

$$(m+n)^2\alpha^2+m^2\beta^2+n^2\gamma^2-2mn\beta\gamma+2(m+n)(n\alpha\gamma+m\alpha\beta)=0,$$

or $\quad m^2(\alpha+\beta)^2+n^2(\alpha+\gamma)^2+2mn(\alpha^2+\alpha\beta+\alpha\gamma-\beta\gamma)=0,$

or, in virtue of (3),

$$m^2(\alpha+\beta)(\gamma+\delta)+n^2(\alpha+\gamma)(\beta+\delta)+2mn(\alpha\delta+\beta\gamma)=0,$$

or, $\quad m^2(\alpha\gamma+\beta\delta)+n^2(\alpha\beta+\gamma\delta)+(m+n)^2(\alpha\delta+\beta\gamma)=0.$

We may write the equation with only one constant, thus:

$$(\alpha\beta+\gamma\delta)+\kappa^2(\alpha\gamma+\beta\delta)+(1+\kappa)^2(\alpha\delta+\beta\gamma)=0\ldots(4).$$

But it is more symmetrical to express it thus,

$$\lambda(\alpha\beta+\gamma\delta)+\mu(\alpha\gamma+\beta\delta)+\nu(\alpha\delta+\beta\gamma)=0\ldots\ldots(5),$$

$\lambda$, $\mu$, $\nu$ being understood to be subject to the relation

$$\sqrt{(\lambda)}+\sqrt{(\mu)}+\sqrt{(\nu)}=0\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

8. It may be shewn, as in trilinear coordinates, that the polar of any point $(\alpha', \beta', \gamma', \delta')$ with respect to the conic whose equation is $f(\alpha, \beta, \gamma, \delta)=0$ is represented by the equation

$$\alpha'\frac{df}{d\alpha}+\beta'\frac{df}{d\beta}+\gamma'\frac{df}{d\gamma}+\delta'\frac{df}{d\delta}=0.$$

If $(\alpha', \beta', \gamma', \delta')$ lie on the conic, of course this equation will represent the tangent at the point.

9. As an example we may prove the well known theorem:

*Given four points on a conic, the polar of any fixed point passes through a fixed point.*

Take the four given points as the angular points of the quadrilateral of reference.

Then the equation to the conic may be written

$$\alpha\gamma - \kappa\beta\delta = 0.$$

Let $\alpha', \beta', \gamma', \delta'$ be the coordinates of the fixed point, then the polar is given by the equation

$$\gamma'\alpha + \alpha'\gamma - \kappa(\delta'\beta + \beta'\delta) = 0,$$

and therefore always passes through the point determined by

$$\frac{\alpha}{\alpha'} + \frac{\gamma}{\gamma'} = 0, \quad \frac{\beta}{\beta'} + \frac{\delta}{\delta'} = 0,$$

a fixed point since $\alpha', \gamma', \beta', \delta'$ are constant. Q.E.D.

10. The coordinates of the angular point $A$ are given by

$$\alpha = \delta = 0, \quad \beta = -\gamma = \frac{1}{B - C},$$

and we obtain similiar equations for the coordinates of $B, C, D$.

Hence, considering the circumscribing conic $(\alpha\gamma - \kappa\beta\delta = 0)$, we obtain as the equation to the

tangent at $A$, $\alpha + \kappa\delta = 0$,
tangent at $B$, $\alpha + \kappa\beta = 0$,
tangent at $C$, $\gamma + \kappa\beta = 0$,
tangent at $D$, $\gamma + \kappa\delta = 0$.

Whence we observe that in any conic passing through the four points $A, B, C, D$

the tangents at $B$ and $C$, $D$ and $A$ meet on $OE$,
$A$ and $B$, $C$ and $D$ meet on $OF$,
$A$ and $C$, $B$ and $D$ meet on $EF$.

A great number of well-known properties follow from these results: we will enunciate two of them.

I. *If a quadrilateral be inscribed in a conic section, and another quadrilateral be described touching the conic in the angular points of the former one, the four interior diagonals of the two quadrilaterals meet in one point* ($O$), *and the two exterior diagonals coincide* ($EF$).

II. *If a quadrilateral be inscribed in a conic, the points of intersection of opposite sides and the points of intersection of tangents at opposite angles lie all in one straight line.* Camb. Math. Tripos Examination, 1847.

11. In the next article the angles between the lines of reference will be involved. We shall denote by $A$, $B$, $C$, $D$ the angles between the consecutive sides $d$ and $a$, $a$ and $b$, $b$ and $c$, $c$ and $d$ respectively; and by $E$, $F$ the angles between the opposite sides $a$ and $c$, $b$ and $d$, respectively, the interior angles towards the quadrilateral $ABCD$ being taken in each case.

12. *To shew that the circles circumscribing the triangles $FDC$, $EAD$, $FAB$, and $EBC$ pass all through one point.*

Using perpendicular coordinates, the equations to the four circles are known to be

$$\frac{\sin D}{\beta} - \frac{\sin F}{\gamma} + \frac{\sin C}{\delta} = 0,$$

$$\frac{\sin D}{\alpha} + \frac{\sin A}{\gamma} + \frac{\sin E}{\delta} = 0,$$

$$\frac{\sin F}{\alpha} + \frac{\sin A}{\beta} + \frac{\sin B}{\delta} = 0,$$

$$\frac{\sin C}{\alpha} - \frac{\sin E}{\beta} + \frac{\sin B}{\gamma} = 0,$$

and these will be satisfied by the same values of $\alpha$, $\beta$, $\gamma$, $\delta$, provided

$$\begin{vmatrix} 0, & \sin D, & -\sin F, & \sin C, \\ \sin D, & 0, & \sin A, & \sin E, \\ \sin F, & \sin A, & 0, & \sin B, \\ \sin C, & -\sin E, & \sin B, & 0, \end{vmatrix} = 0 \ldots (1).$$

Now, projecting the sides of the closed figure $ABCD$ upon lines at right angles to its four sides in order, we get

$$\begin{aligned} b\sin B + c\sin E - d\sin A &= 0,\\ -a\sin B + c\sin C + d\sin F &= 0,\\ a\sin E + b\sin C - d\sin D &= 0,\\ -a\sin A + b\sin F + c\sin D &= 0,\end{aligned}$$

whence eliminating $a : b : c : d$,

$$\begin{vmatrix} 0, & \sin B, & \sin E, & -\sin A \\ -\sin B, & 0, & \sin C, & \sin F \\ \sin E, & \sin C, & 0, & -\sin D \\ -\sin A, & \sin F, & \sin D, & 0 \end{vmatrix} = 0 \;\ldots\ldots\; (2).$$

The equations (1) and (2) are the same, each of the determinants being

$$(\sin E \sin F + \sin A \sin C - \sin B \sin D)^2.$$

Hence the condition (1) is satisfied, and therefore the four circles meet in a point. Q.E.D.

13. *To find the condition that the general equation of the second degree*

$$\lambda\alpha\beta + \lambda'\gamma\delta + \mu\alpha\gamma + \mu'\beta\delta + \nu\alpha\delta + \nu'\beta\gamma = 0 \;\ldots\ldots\; (1)$$

*should represent a parabola.*

Returning to the coordinates of Art. 2, the tangent at any point $(\alpha', \beta', \gamma', \delta')$, is given by

$$\alpha(\lambda\beta' + \mu\gamma' + \nu\delta') + \beta(\lambda\alpha' + \nu'\gamma' + \mu'\delta') + \gamma(\mu\alpha' + \nu'\beta' + \lambda'\delta') + \delta(\nu\alpha' + \mu'\beta' + \lambda'\gamma') = 0.$$

Hence, if the line at infinity be a tangent, we must have

$$\frac{\lambda\beta' + \mu\gamma' + \nu\delta'}{A+K} = \frac{\lambda\alpha' + \nu'\gamma' + \mu'\delta'}{B+K} = \frac{\mu\alpha' + \nu'\beta' + \lambda'\delta'}{C+K} = \frac{\nu\alpha' + \mu'\beta' + \lambda'\gamma'}{D+K} = -x, \text{ suppose;}$$

therefore

$$\begin{aligned} \lambda\beta' + \mu\gamma' + \nu\delta' + Ax + Kx &= 0,\\ \lambda\alpha' + \nu'\gamma' + \mu'\delta' + Bx + Kx &= 0,\\ \mu\alpha' + \nu'\beta' + \lambda'\delta' + Cx + Kx &= 0,\\ \nu\alpha' + \mu'\beta' + \lambda'\gamma' + Dx + Kx &= 0,\end{aligned}$$

also, since the point of contact $(\alpha', \beta', \gamma', \delta')$ must be on the line at infinity,

$$A\alpha' + B\beta' + C\gamma' + D\delta' = 0,$$
$$\alpha' + \beta' + \gamma' + \delta' = 0.$$

Eliminating from the last six equations the five ratios

$$\alpha' : \beta' : \gamma' : \delta' : x : Kx,$$

we get

$$\begin{vmatrix} 0, & 1, & 1, & 1, & 1, & 0 \\ 1, & 0, & \lambda, & \mu, & \nu, & A \\ 1, & \lambda, & 0, & \nu', & \mu', & B \\ 1, & \mu, & \nu', & 0, & \lambda', & C \\ 1, & \nu, & \mu', & \lambda', & 0, & D \\ 0, & A, & B, & C, & D, & 0 \end{vmatrix} = 0,$$

which is therefore the condition necessary and sufficient that the conic represented by the general equation (1), should be a parabola.

14. The condition obtained in the last article, reduces in the case of the circumscribed conic

$$\alpha\gamma - \kappa\beta\delta = 0,$$

to the form

$$\kappa^2 (A - C)^2 - \kappa \{(A - B)^2 + (C - D)^2 + (A - D)^2 + (B - C)^2\} + (B - D)^2 = 0,$$

a quadratic equation to determine $\kappa$. According as the roots of this equation are real or unreal, there can be drawn no parabola, or two and only two parabolas through the four points $A$, $B$, $C$, $D$. Hence, if one parabola can be drawn through four given points, there can generally be another also drawn through them.

The two roots of the equation which gives $\kappa$, cannot be equal, unless either

$$A = B \text{ and } C = D,$$

or

$$A = D \text{ and } C = B,$$

that is, they cannot be equal unless two adjacent angles of the quadrilateral of reference be equal, and the other two equal and supplementary to the former.

Hence, *through any four points on a parabola another parabola can always be drawn, unless the four points be symmetrically situated at the extremities of two parallel chords perpendicular to the axis.*

15. In the case of the inscribed conic

$$\lambda\,(\alpha\beta+\gamma\delta)+\mu\,(\alpha\gamma+\beta\delta)+\nu\,(\alpha\delta+\beta\gamma)=0 \;\ldots\ldots\; (1),$$

the condition of Art. 9 becomes

$$\begin{vmatrix} 0, & 1, & 1, & 1, & 1, & 0 \\ 1, & 0, & \lambda, & \mu, & \nu, & A \\ 1, & \lambda, & 0, & \nu, & \mu, & B \\ 1, & \mu, & \nu, & 0, & \lambda, & C \\ 1, & \nu, & \mu, & \lambda, & 0, & D \\ 0, & A, & B, & C, & D, & 0 \end{vmatrix}=0,$$

which reduces to

$$\frac{\{(A+B)-(C+D)\}^2}{\mu+\nu-\lambda}+\frac{\{(A+C)-(B+D)\}^2}{\nu+\lambda-\mu}$$
$$+\frac{\{(A+D)-(B+C)\}^2}{\lambda+\mu-\nu}=0 \;\ldots\ldots\ldots\; (2).$$

But since (Art. 7),

$$\sqrt{(\lambda)}+\sqrt{(\mu)}+\sqrt{(\nu)}=0 \;\ldots\ldots\ldots\ldots\ldots\; (3),$$

we have

$$(\mu+\nu-\lambda)^2=4\mu\nu,$$
$$(\nu+\lambda-\mu)^2=4\nu\lambda,$$
$$(\lambda+\mu-\nu)^2=4\lambda\mu.$$

Hence (2) becomes

$$\{(A+B)-(C+D)\}^2\sqrt{(\lambda)}+\{(A+C)-(B+D)\}^2\sqrt{(\mu)}$$
$$+\{(A+D)-(B+C)\}\sqrt{(\nu)}=0 \;\ldots\ldots\ldots\; (4).$$

From (3) and (4), we get

$$\frac{\sqrt{(\lambda)}}{AC+BD-AD-BC}=\frac{\sqrt{(\mu)}}{AD+BC-AB-CD}$$
$$=\frac{\sqrt{(\nu)}}{AB+CD-AB-BD},$$

or
$$\frac{\sqrt{(\lambda)}}{(A-B)(C-D)}=\frac{\sqrt{(\mu)}}{(A-C)(D-B)}=\frac{\sqrt{(\nu)}}{(A-D)(B-C)}.$$

Hence there can be but one parabola inscribed in the quadrilateral, and its equation is

$$(A-B)^2(C-D)^2(\alpha\beta+\gamma\delta)+(A-C)^2(B-D)^2(\alpha\gamma+\beta\delta)$$
$$+(A-D)^2(B-C)^2(\alpha\delta+\beta\gamma)=0.$$

(We know from geometry that the point treated of in Art. 12, is the focus of this parabola.)

*(To be continued.)*

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## QUATERNIONS.

By *P. G. Tait, M.A.*

*(Continued from page* 156.)

XXIII. The general scalar equation of the second order in a vector $\rho$ must evidently contain a term independent of $\rho$, terms of the form $S.a\rho b$ involving $\rho$ to the first degree, and others of the form $S.a\rho b\rho c$ involving $\rho$ to the second degree, $a$, $b$, $c$, &c. being constant quaternions. Now the term $S.a\rho b$ may be written

$$S.(Sa+Va)\rho(Sb+Vb),$$

or, expanding and attending to (IX., X.),

$$SaS\rho Vb+SbS\rho Va+S.\rho VbVa,$$

each of which may evidently be put in the form $S\epsilon\rho$, where $\epsilon$ is a known vector.

Similarly the term $S.a\rho b\rho c$ may be reduced to a set of terms, each of which has one of the forms

$$A\rho^2,\ (S\alpha\rho)^2,\ S\alpha\rho S\beta\rho,$$

the second being merely a particular case of the third. Thus (the numerical factors 2 being introduced for convenience) we may write the general scalar equation of the second degree as follows:

$$2\Sigma.S\alpha\rho S\beta\rho+A\rho^2+2S\gamma\rho=C\ldots\ldots\ldots\ldots(1).$$

Change the origin to $D$ where $OD=\delta$, then $\rho$ becomes $\rho+\delta$, and the equation takes the form

$$2\Sigma . S\alpha\rho S\beta\rho + A\rho^2 + 2\Sigma . (S\alpha\rho S\beta\delta + S\beta\rho S\alpha\delta) + 2AS\delta\rho + 2S\gamma\rho$$
$$+ 2\Sigma . S\alpha\delta S\beta\delta + A\delta^2 + 2S\gamma\delta - C = 0;$$

from which the first power of $\rho$ disappears if

$$\Sigma(\alpha S\beta\delta + \beta S\alpha\delta) + A\delta + \gamma = 0 \ldots\ldots\ldots\ldots (2),$$

a vector equation of the first degree, which in general gives a single definite value for $\delta$. This we assume for the present; and will explain, with its exceptions, when we come to the solution of linear equations generally.

With this value of $\delta$, and putting

$$D = C - 2S\gamma\delta - A\delta^2 - 2\Sigma . S\alpha\delta S\beta\delta,$$

the equation becomes

$$2\Sigma . S\alpha\rho S\beta\rho + A\rho^2 = D.$$

If $D=0$, the surface is conical (XXII.), if not, it is an ellipsoid or hyperboloid. Unless expressly stated not to be, the surface will, when $D$ is not zero, be considered an ellipsoid. By this we avoid for the time some rather delicate considerations.

By dividing by $D$ (when the latter is finite) and thus altering only the tensors of the constants, we see that the equation to central surfaces of the second order, referred to the centre, is

$$2\Sigma(S\alpha\rho S\beta\rho) + g\rho^2 = 1 \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

{Or $=0$, but for the present we leave the latter case, that of cones, which indeed the reader may supply for himself, as it simply involves the disappearance of the right-hand member of the equation.}

XXIV. Differentiating, we obtain

$$2\Sigma\{S\alpha d\rho S\beta\rho + S\alpha\rho S\beta d\rho\} + 2gS\rho d\rho = 0,$$

or
$$S.d\rho\{\Sigma(\alpha S\beta\rho + \beta S\alpha\rho) + g\rho\} = 0,$$

and therefore, by (XXII.), the tangent plane is

$$S(\varpi - \rho)\{\Sigma(\alpha S\beta\rho + \beta S\alpha\rho) + g\rho\} = 0,$$

*i.e.*, $S.\varpi\{\Sigma(\alpha S\beta\rho + \beta S\alpha\rho) + g\rho\} = 1$, by (XXIII., 3).

Hence if $\nu = \Sigma (\alpha S\beta\rho + \beta S\alpha\rho) + g\rho$

the tangent plane is $S\nu\varpi = 1$,

and the surface itself is $S\nu\rho = 1$.

And, as $\nu^{-1}$ is evidently the perpendicular on the tangent plane, $\nu$ is called the *vector of proximity.*

XXV. Sir W. R. Hamilton uses for $\nu$, which is evidently a linear and vector function of $\rho$, the notation $\phi(\rho)$, $\phi$ expressing a functional operation. Hence

$$\phi(\rho) = \Sigma (\alpha S\beta\rho + \beta S\alpha\rho) + g\rho.$$

With this definition of $\phi$, it is easy to see that

(*a*) $\phi(\rho + \sigma) = \phi\rho + \phi\sigma$, &c., for *any* two or *more* vectors.

(*b*) $\phi(x\rho) = x\phi\rho$, a particular case of (*a*), $x$ being a scalar.

(*c*) $d\phi\rho = \phi(d\rho)$.

(*d*) $S\sigma\phi\rho = \Sigma (S\alpha\sigma S\beta\rho + S\beta\sigma S\alpha\rho) + gS\rho\sigma = S\rho\phi\sigma$.

This last property is of great importance.

XXVI. Thus the general equation may now be written

$$S\rho\phi\rho = 1 \text{ ........................ } (1).$$

Differentiating, $S d\rho\phi\rho + S\rho d\phi\rho = 0$,

which, by applying (*c*) and then (*d*) to the last term on the left, gives

$$2 S d\rho\phi\rho = 0,$$

and therefore, as in (XXIV.), though now much more simply, *the tangent plane at the extremity of* $\rho$ is

$$S(\varpi - \rho)\phi\rho = 0,$$

or $S\varpi\phi\rho = S\rho\phi\rho = 1$.

If this pass through $A$, $(OA = \alpha)$, we have

$$S\alpha\phi\rho = 1,$$

or, by (*d*), $S\rho\phi\alpha = 1$,

for all possible points of contact.

This is therefore the *equation to the plane of contact of tangent planes drawn from A.*

*To find the enveloping cone whose vertex is $A$*, notice that

$$(S\rho\phi\rho - 1) + p\,(S\rho\phi\alpha - 1)^2 = 0,$$

is the equation to a surface of the second order *touching* the ellipsoid along its intersection with the plane. If this pass through $A$ we have

$$(S\alpha\phi\alpha - 1) + p\,(S\alpha\phi\alpha - 1)^2 = 0,$$

and $p$ is found. Then our equation becomes

$$(S\rho\phi\rho - 1)\,(S\alpha\phi\alpha - 1) - (S\rho\phi\alpha - 1)^2 = 0,$$

which is the cone required. To assure ourselves of this, transfer the origin to $A$, by putting $\rho + \alpha$ for $\rho$. The result is, using ($a$) and ($d$),

$$(S\rho\phi\rho + 2S\rho\phi\alpha + S\alpha\phi\alpha - 1)\,(S\alpha\phi\alpha - 1) - (S\rho\phi\alpha + S\alpha\phi\alpha - 1)^2 = 0,$$

or

$$S\rho\phi\rho\,(S\alpha\phi\alpha - 1) - (S\rho\phi\alpha)^2 = 0,$$

which is homogeneous in $T\rho^2$, and is therefore the equation to a cone, (XXII.).

Suppose $A$ infinitely distant, then we may put $x\alpha$ for $\alpha$, where $x$ is indefinitely great, and, omitting all but the higher terms, the equation to the cylinder formed by tangent lines parallel to $\alpha$ is

$$(S\rho\phi\rho - 1)\,S\alpha\phi\alpha - (S\rho\phi\alpha)^2 = 0.$$

XXVII. To study the nature of the surface more closely, let us *find the locus of the middle points of a system of parallel chords.*

Let them be parallel to $\alpha$, then, if $\varpi$ be the vector of the middle point of one of them, $\varpi + x\alpha$ and $\varpi - x\alpha$ are simultaneous values of $\rho$ which ought to satisfy XXVI., 1.

That is

$$S.(\varpi \pm x\alpha)\,\phi\,(\varpi \pm x\alpha) = 1.$$

Hence, by ($a$) and ($d$), as before,

$$S\varpi\phi\varpi + x^2 S\alpha\phi\alpha = 1,$$

$$S.\varpi\phi\alpha = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

The latter equation shows that the locus of the extremity of $\varpi$, the middle point of a chord parallel to $\alpha$, is a plane through the centre, whose normal is $\phi\alpha$; that is, a plane parallel to the tangent plane at the point where $OA$ cuts the surface. And ($d$) shows that this relation is reciprocal—so that if $\beta$ be *any* value of $\varpi$, *i.e.*, be any vector in the plane (1), $\alpha$ will be a vector in a diametral plane which bisects all chords parallel to $\beta$. The equations to these planes are

$$S.\varpi\phi\alpha = 0,$$
$$S.\varpi\phi\beta = 0,$$

so that if $V.\phi\alpha\phi\beta = \gamma$ (suppose) is their line of intersection, we have

$$\left.\begin{array}{l} S\gamma\phi\alpha = 0 = S\alpha\phi\gamma \\ S\gamma\phi\beta = 0 = S\beta\phi\gamma \\ S\beta\phi\alpha = 0 = S\alpha\phi\beta \end{array}\right\} \text{......................(2)},$$

and (1) gives

Hence there is *an infinite number of sets of three vectors* $\alpha$, $\beta$, $\gamma$, *such that all chords parallel to any one are bisected by the diametral plane containing the other two.*

XXVIII. It is worthy of notice that, by (XXVII., 2),

$$\phi\gamma \parallel V\alpha\beta,$$

so that

$$\phi V\phi\alpha\phi\beta = m V\alpha\beta,$$

where $m$ is a scalar, and it will easily be seen that this equation is true *without* the restriction

$$S\alpha\phi\beta = 0,$$

and is therefore true of *any* two vectors whatever. Let $\gamma$ be *any* third vector, and operate on each side by $S.\gamma$. Remembering ($d$) in transforming the left-hand side, we have

$$S.\phi\alpha\phi\beta\phi\gamma = mS\alpha\beta\gamma \text{..................(1).}$$

which determines $m$, and the nature of the process shows that $m$ cannot involve $\alpha$, $\beta$, or $\gamma$, but *must depend entirely on the form and constitutents of the function* $\phi$. $m$ thus determined, we find by operating by the inverse of $\phi$

$$m\phi^{-1}(V\alpha\beta) = V\phi\alpha\phi\beta \text{..................(2).}$$

This process and its results, due to Sir W. R. Hamilton, are of the utmost value, as we shall afterwards see. Meanwhile, as they have been somewhat irregularly introduced here and merely because the equations on which they depend happened to have been written down, we leave them with the single remark that they give us the means of inverting the linear and vector function $\phi$, *i.e.*, of *solving the general vector equation of the first degree* $\phi(\rho) = \delta$, *by direct operations.*

XXIX. It is evident from (XI., 4) that any vector may be expressed as a linear function of any three others not in the same plane, let then

$$\rho = x\alpha + y\beta + z\gamma,$$

where, by (XXVII., 2),

$$S\alpha\phi\beta = S\beta\phi\alpha = 0,$$
$$S\alpha\phi\gamma = S\gamma\phi\alpha = 0,$$
$$S\beta\phi\gamma = S\gamma\phi\beta = 0.$$

And let

$$\left.\begin{aligned} S\alpha\phi\alpha &= 1 \\ S\beta\phi\beta &= 1 \\ S\gamma\phi\gamma &= 1 \end{aligned}\right\},$$

so that $\alpha$, $\beta$, and $\gamma$ are vector conjugate semidiameters.

Substituting the above value of $\rho$ in the equation to the surface, and attending to the equations in $\alpha$, $\beta$, $\gamma$ and to ($a$), ($b$), and ($d$),

$$\begin{aligned} S\rho\phi\rho &= S(x\alpha + y\beta + z\gamma)\,\phi\,(x\alpha + y\beta + z\gamma), \\ &= x^2 + y^2 + z^2 = 1. \end{aligned}$$

To transform this equation to Cartesian coordinates, we notice that $x$ is the ratio which the projection of $\rho$ on $\alpha$ bears to $\alpha$ itself, &c. If therefore we take the conjugate diameters as axes of $\xi$, $\eta$, $\zeta$, and their lengths as $a$, $b$, $c$, the above equation becomes at once

$$\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} + \frac{\zeta^2}{c^2} = 1,$$

the ordinary equation to the ellipsoid referred to conjugate diameters.

XXX. If we write $-\psi^2$ instead of $\phi$, these equations assume an interesting form. We take for granted, what we shall afterwards show, that this halving or extracting the root of the vector function is lawful, and that the new linear and vector function has the same properties (*a*), (*b*), (*c*), (*d*) as the old. The equation to the surface now becomes

$$S\rho\psi^2\rho = -1,$$

or
$$S\psi\rho\psi\rho = -1,$$

or, finally,
$$T\psi\rho = 1.$$

If we compare this with the equation to the unit-sphere $T\rho = 1$, we see at once the analogy between the two surfaces. *The sphere can be changed into the ellipsoid, or vice versâ, by a linear deformation of each vector, the operator being the function* $\psi$ *or its inverse.*

Equations (2, XXVII.) now become

$$S\alpha\psi^2\beta = 0 = S\psi\alpha\psi\beta, \text{ \&c.} \dots\dots\dots\dots\dots (1),$$

so that $\psi\alpha$, $\psi\beta$, $\psi\gamma$, *the vectors of the unit-sphere which correspond to semiconjugate diameters of the ellipsoid, form a rectangular system.*

We may remark here, that, as the equation to the ellipsoid referred to its principal axes is a case of (XXIX.), we may now suppose $i$, $j$, and $k$ to have these directions, and the equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, which, in quaternions, is

$$S\rho\phi\rho = \frac{(Si\rho)^2}{a^2} + \frac{(Sj\rho)^2}{b^2} + \frac{(Sk\rho)^2}{c^2} = 1.$$

We here tacitly assume the existence of such axes, but, by the help of Sir W. R. Hamilton's method, hinted at in (XXVIII.), we at once arrive at the cubic equation which gives them. As the author has not yet published his results in their new and exceedingly simple form, we refer the reader to the investigation in his *Lectures*, p. 559, the transformations in which will prove an excellent exercise.

It is evident from the last written equation that

$$\phi\rho = +\frac{iSi\rho}{a^2} + \frac{jSj\rho}{b^2} + \frac{kSk\rho}{c^2},$$

and
$$\psi\rho = -\left(\frac{iSi\rho}{a} + \frac{jSj\rho}{b} + \frac{kSk\rho}{c}\right),$$

which latter may be easily verified by showing that

$$\psi^2(\rho) = -\phi\rho.$$

And this expression enables us to verify the assertion at the commencement of this section about the properties of $\psi$.

As $Si\rho = -x$, &c., $x$, $y$, $z$ being the Cartesian coordinates referred to the principal axes, we have now the means of at once transforming any quaternion result connected with the ellipsoid into the ordinary one.

XXXI. Before proceeding to other forms of the equation to the ellipsoid, we may use those already given in solving a few problems.

*Find the locus of a point when the perpendicular from the centre on its polar plane is of constant length.*

If $\varpi$ be the vector of the point, the polar plane is

$$S\rho\phi\varpi = 1,$$

and the length of the perpendicular from $O$ is $\frac{1}{T\phi\varpi}$, (XXII.).

Hence the required locus is

$$T\phi\varpi = C,$$

or

$$S\varpi\phi^2\varpi = -C^2,$$

a concentric ellipsoid, with its axes in the same direction as those of the first. By (XXX.) its Cartesian equation is

$$\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} = C^2. \quad \text{(See XXXII.)}$$

*Find the locus of a point whose distance from a given point is always in a given ratio to its distance from a given line.*

Let $\beta$ be the given line, and $A$ ($OA = \alpha$) the given point, and let $S\alpha\beta = 0$. Then for any one of the required points

$$T(\rho - \alpha) = eTV\beta\rho,$$

a surface of the second order, which may be written

$$\rho^2 - 2S\alpha\rho + \alpha^2 = e^2(S^2\beta\rho - \beta^2\rho^2).$$

Let the centre be at $\delta$, then, by XXIII.,

$$\rho^2 + 2S\rho(\delta - \alpha) + (\delta - \alpha)^2 = e^2\{S^2.\beta(\rho + \delta) - \beta^2(\rho + \delta)^2\},$$

and, that the first power of $\rho$ may disappear,

$$(\delta - \alpha) = e^2(\beta S\beta\delta - \beta^2\delta),$$

a linear equation for $\delta$. To solve it, note that $S\alpha\beta = 0$, operate by $S.\beta$ and we get

$$S\beta\delta = 0.$$

Hence $$\delta - \alpha = -e^2\beta^2\delta,$$

or $$\delta = \frac{\alpha}{1 + e^2\beta^2}.$$

Referred to this point the equation becomes

$$(1 + e^2\beta^2)\,\rho^2 - e^2S^2\beta\rho + \frac{e^4\beta^4\alpha^2}{(1 + e^2\beta^2)^2} = 0,$$

which shows that it belongs to a surface of revolution whose axis is parallel to $\beta$, as its intersection with a plane $S\beta\rho = a$ perpendicular to that axis lies also on the sphere

$$\rho^2 = \frac{e^2a^2}{1 + e^2\beta^2} - \frac{e^4\beta^4\alpha^2}{(1 + e^2\beta^2)^3}.$$

*A sphere, passing through the centre of an ellipsoid, is cut by a series of spheres whose centres are on the ellipsoid and which pass through the centre thereof; find the envelop of the planes of intersection.*

Let $(\rho - \alpha)^2 = \alpha^2$ be the first sphere, *i.e.*,

$$\rho^2 - 2S\alpha\rho = 0.$$

One of the others is

$$\rho^2 - 2S\varpi\rho = 0,$$

where $$S\varpi\phi\varpi = 1.$$

The plane of intersection is

$$S(\varpi - \alpha)\,\rho = 0.$$

Hence, for the envelop,

$$\left.\begin{aligned} S\varpi'\phi\varpi &= 0, \\ S\varpi'\rho &= 0, \end{aligned}\right\}$$

or $$\phi\varpi = x\rho,\quad \{Vx = 0\},$$

*i.e.*, $$\varpi = x\phi^{-1}\rho.$$

Hence $$\left.\begin{aligned} x^2S\rho\phi^{-1}\rho &= 1, \\ xS\rho\phi^{-1}\rho &= S\alpha\rho, \end{aligned}\right\}$$
and

and, eliminating $x$,

$$S\rho\phi^{-1}\rho = (S\alpha\rho)^2,$$

a cone of the second order.

*From a point in the outer of two concentric ellipsoids a tangent cone is drawn to the inner, find the envelop of the plane of contact.*

If $S\varpi\phi\varpi = 1$ be the outer, and $S\rho\psi\rho = 1$ be the inner, the plane of contact is

$$S\varpi\psi\rho = 1.$$

Hence, for the envelop,

$$S\varpi'\psi\rho = 0,$$

$$S\varpi'\phi\varpi = 0,$$

therefore $$\phi\varpi = x\psi\rho,$$

or $$\varpi = x\phi^{-1}\psi\rho.$$

This gives and $$\left.\begin{aligned} xS.\psi\rho\phi^{-1}\psi\rho &= 1, \\ x^2S.\psi\rho\phi^{-1}\psi\rho &= 1, \end{aligned}\right\}$$

and therefore, eliminating $x$,

$$S.\psi\rho\phi^{-1}\psi\rho = 1,$$

or $$S.\rho\psi^{-1}\phi\psi\rho = 1,$$

another concentric ellipsoid, as $\psi^{-1}\phi\psi$ is a linear and vector function $=\chi$ suppose, so that the equation may be written

$$S\rho\chi\rho = 1.$$

*Find the locus of intersection of tangent planes at the extremities of conjugate diameters.*

If $\alpha$, $\beta$, $\gamma$ be the vector semi-diameters, the planes are

$$\left.\begin{aligned} S\varpi\psi^2\alpha &= -1, \\ S\varpi\psi^2\beta &= -1, \\ S\varpi\psi^2\gamma &= -1, \end{aligned}\right\}$$

with the conditions (XXX., 1).

Hence

$$-\psi\varpi S.\psi\alpha\psi\beta\psi\gamma = \psi\varpi = \psi\alpha + \psi\beta + \psi\gamma, \text{ by (XI.)},$$

therefore $$T\psi\varpi = \sqrt{(3)},$$

since $\psi\alpha$, $\psi\beta$, $\psi\gamma$ form a rectangular system of unit vectors.

This may also evidently be written

$$S\varpi\psi^2\varpi = -3,$$

showing that the locus is similar and similiarly situated to the given ellipsoid, but larger in the ratio $\sqrt{(3)} : 1$.

*Find the locus of the intersection of three spheres whose diameters are semi-conjugate diameters of an ellipsoid.*

If $\alpha$ be one of the semi-conjugate diameters

$$S\alpha\psi^2\alpha = -1.$$

And the corresponding sphere is (XXI.)

$$\rho^2 - S\alpha\rho = 0,$$

or

$$\rho^2 - S\psi\alpha\psi^{-1}\rho = 0,$$

with similar equations in $\beta$ and $\gamma$. Hence, by (XI.),

$$\psi^{-1}\rho S.\psi\alpha\psi\beta\psi\gamma = -\psi^{-1}\rho = \rho^2(\psi\alpha + \psi\beta + \psi\gamma),$$

and, taking tensors,

$$T\psi^{-1}\rho = \sqrt{(3)}\, T\rho^2,$$

or

$$T\psi^{-1}\rho^{-1} = \sqrt{(3)},$$

or, finally,

$$S\rho\psi^{-2}\rho = -3\rho^4.$$

This is Fresnel's *Surface of Elasticity in the Undulatory Theory.*

XXXII. Before going farther we may prove some useful properties of the function $\phi$ in the form

$$\phi\rho = \frac{iSi\rho}{a^2} + \frac{jSj\rho}{b^2} + \frac{kSk\rho}{c^2}.$$

Since $\rho = -iSi\rho - jSj\rho - kSk\rho$, (XI.) ............(1),

it is evident that

$$\phi i = -\frac{i}{a^2}, \quad \phi j = -\frac{j}{b^2}, \quad \phi k = -\frac{k}{c^2}.$$

Hence

$$\phi^2\rho = -\frac{iSi\rho}{a^4} - \frac{jSj\rho}{b^4} - \frac{kSk\rho}{c^4}.$$

Also

$$\phi^{-1}\rho = a^2 iSi\rho + b^2 jSj\rho + c^2 kSk\rho,$$

and so on.

Again, if $\alpha$, $\beta$, $\gamma$ be *any* rectangular unit vectors

$$S\alpha\phi\alpha = \frac{(Si\alpha)^2}{a^2} + \frac{(Sj\alpha)^2}{b^2} + \frac{(Sk\alpha)^2}{c^2},$$

&c. = &c.

And as $(Si\rho)^2 + (Sj\rho)^2 + (Sk\rho)^2 = -\rho^2$ (by taking tensors in 1),

$$S\alpha\phi\alpha + S\beta\phi\beta + S\gamma\phi\gamma = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}.$$

Again,

$$S.\phi\alpha\phi\beta\phi\gamma = S\left(\frac{iSi\alpha}{a^2}+\ldots\right)\left(\frac{iSi\beta}{a^2}+\ldots\right)\left(\frac{iSi\gamma}{a^2}+\ldots\right)$$

$$= -\begin{vmatrix} \frac{Si\alpha}{a^2}, & \frac{Sj\alpha}{b^2}, & \frac{Sk\alpha}{c^2} \\ \frac{Si\beta}{a^2}, & \frac{Sj\beta}{b^2}, & \frac{Sk\beta}{c^2} \\ \frac{Si\gamma}{a^2}, & \frac{Sj\gamma}{b^2}, & \frac{Sk\gamma}{c^2} \end{vmatrix} = \frac{-1}{a^2b^2c^2}\begin{vmatrix} Si\alpha, & Sj\alpha, & Sk\alpha \\ Si\beta, & Sj\beta, & Sk\beta \\ Si\gamma, & Sj\gamma, & Sk\gamma \end{vmatrix} = \frac{1}{a^2b^2c^2}.$$

And so on.

*Find the locus of intersection of a rectangular system of three tangents to an ellipsoid.*

If $\varpi$ be the vector of the point of intersection, $\alpha$, $\beta$, $\gamma$ the tangents; then, since $\varpi + x\alpha$ should give equal values of $x$ when substituted in the equation to the surface, we have

$$S(\varpi + x\alpha)\,\phi\,(\varpi + x\alpha) = 1,$$

or

$$x^2 S\alpha\phi\alpha + 2xS\varpi\phi\alpha + (S\varpi\phi\varpi - 1) = 0,$$

and, therefore, $(S\varpi\phi\alpha)^2 = S\alpha\phi\alpha\,(S\varpi\phi\varpi - 1)$.

Adding this to the two similar equations in $\beta$ and $\gamma$

$$(S\alpha\phi\varpi)^2 + (S\beta\phi\varpi)^2 + (S\gamma\phi\varpi)^2$$
$$= (S\alpha\phi\alpha + S\beta\phi\beta + S\gamma\phi\gamma)\,(S\varpi\phi\varpi - 1),$$

or

$$-(\phi\varpi)^2 = \left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)(S\varpi\phi\varpi - 1),$$

or

$$S.\varpi\left\{\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\phi + \phi^2\right\}\varpi = \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2},$$

an ellipsoid concentric with the first, and having the directions of its axes coincident with those of the former.

*If a rectangular system of chords be drawn through any point within an ellipsoid, the sum of the reciprocals of the rectangles under the segments into which they are divided is constant.*

With the notation of the solution of the preceding problem, $\varpi$ giving the intersection of the vectors, it is evident that the product of the values of $x$ is minus one of the rectangles in question.

Hence the required sum is $\Sigma\left\{\frac{-S\alpha\phi\alpha}{S\varpi\phi\varpi-1}\right\}$,

or $$-\frac{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}{S\varpi\phi\varpi-1};$$

and it is evident that this depends only on $S\varpi\phi\varpi$, hence the sum is unaltered if the point be taken anywhere on the surface of a similar, similarly situated, and concentric, ellipsoid.

*Show that if three rectangular vectors be drawn from a point of an ellipsoid, the plane containing their other extremities passes through a fixed point. Find the locus of the latter as the former varies.*

With the same notation as before, we have

$$S\varpi\phi\varpi=1,$$

$$S(\varpi+x\alpha)\,\phi\,(\varpi+x\alpha)=1,$$

therefore $$x=-\frac{2S\alpha\phi\varpi}{S\alpha\phi\alpha}.$$

Hence the required plane passes through the extremity of

$$\varpi-2\alpha\frac{S\alpha\phi\varpi}{S\alpha\phi\alpha},$$

and those of two other lines similarly determined. Its equation is therefore

$$S.\left(\rho-\varpi+2\alpha\frac{S\alpha\phi\varpi}{S\alpha\phi\alpha}\right)\left(\frac{\alpha S\alpha\phi\varpi}{S\alpha\phi\alpha}-\frac{\beta S\beta\phi\varpi}{S\beta\phi\beta}\right)\left(\frac{\beta S\beta\phi\varpi}{S\beta\phi\beta}-\frac{\gamma S\gamma\phi\varpi}{S\gamma\phi\gamma}\right)=0,$$

or $$S(\rho-\varpi)\left\{\frac{\gamma S\alpha\phi\varpi S\beta\phi\varpi}{S\alpha\phi\alpha S\beta\phi\beta}+\frac{\beta S\gamma\phi\varpi S\alpha\phi\varpi}{S\gamma\phi\gamma S\alpha\phi\alpha}+\frac{\alpha S\beta\phi\varpi S\gamma\phi\varpi}{S\beta\phi\beta S\gamma\phi\gamma}\right\}$$
$$-2\frac{S\alpha\phi\varpi S\beta\phi\varpi S\gamma\phi\varpi}{S\alpha\phi\alpha S\beta\phi\beta S\gamma\phi\gamma}=0.$$

Let $\rho-\varpi=g\phi\varpi$, therefore

$$g\frac{S\alpha\phi\varpi S\beta\phi\varpi S\gamma\phi\varpi}{S\alpha\phi\alpha S\beta\phi\beta S\gamma\phi\gamma}(S\alpha\phi\alpha+\ldots)-2\frac{S\alpha\phi\varpi S\beta\phi\varpi S\gamma\phi\varpi}{S\alpha\phi\alpha S\beta\phi\beta S\gamma\phi\gamma}=0.$$

or $$g=\frac{2}{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}.$$

Hence, $g$ being thus a definite constant,

$$\rho = \varpi + g\phi\varpi$$

is the vector of a fixed point through which the plane passes.

For the locus of this point as $\varpi$ varies, we have

$$\varpi = (1 + g\phi)^{-1}\rho,$$

and, therefore, by the equation to the surface,

$$S(1 + g\phi)^{-1}\rho\phi(1 + g\phi)^{-1}\rho = 1,$$

or, as it may be written,

$$S.\rho(1 + g\phi)^{-1}\phi(1 + g\phi)^{-1}\rho = 1,$$

a concentric ellipsoid, since $(1 + g\phi)^{-1}\phi(1 + g\phi)^{-1}$ is a linear and vector function, $= \chi$ suppose, and therefore the equation may be written

$$S\rho\chi\rho = 1.$$

For the legality of this proceeding see a subsequent section.

XXXIII. *Find the greatest and least semi-diameters of a diametral section of an ellipsoid.*

We have $S\rho\phi\rho = 1$, and for the cutting plane

$$S\alpha\rho = 0.$$

Also $\qquad T\rho =$ max. or min.

By differentiation we get

$$\left.\begin{aligned} S\rho'\phi\rho &= 0, \\ S\alpha\rho' &= 0, \\ S\rho\rho' &= 0, \end{aligned}\right\}$$

from which, by (XI.),

$$S.\alpha\rho\phi\rho = 0.$$

This, combined with the equations to the plane and ellipsoid, gives the required values of $\rho$. Its interpretation is simply that *the greatest or least diameter of any plane section through the centre, and the normal at its extremity are co-planar with the perpendicular to the section.*

We have now to solve the equations

$$S\alpha\rho = 0,$$

$$S\alpha\rho\phi\rho = 0,$$

$$S\rho\phi\rho = 1.$$

The first two belong to a plane and a cone of the second order, and therefore give in general two directions of intersection. The versors of the two values of $\rho$ being thus determined, the third equation gives the *lengths* of the corresponding diameters. The first two give

$$\phi\rho = xV.\phi^{-1}\alpha V\alpha\rho, \text{ ($x$ being a scalar)}$$

therefore $S\rho\phi\rho = 1 = xS.\rho\,(\rho S\alpha\phi^{-1}\alpha - \alpha S\rho\phi^{-1}\alpha),$

$$= x\rho^2 S\alpha\phi^{-1}\alpha.$$

Hence $$\rho^2\phi\rho = \rho - \frac{\alpha S\rho\phi^{-1}\alpha}{S\alpha\phi^{-1}\alpha},$$

and therefore, separating the operating symbols,

$$(1-\rho^2\phi)\,\rho = \alpha\frac{S\rho\phi^{-1}\alpha}{S\alpha\phi^{-1}\alpha}.$$

Hence $$\rho = \frac{S\rho\phi^{-1}\alpha}{S\alpha\phi^{-1}\alpha}(1-\rho^2\phi)^{-1}\alpha,$$

and, therefore, as $S\alpha\rho = 0$,

$$S.\alpha\,(1-\rho^2\phi)^{-1}\alpha = 0 \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

We have given this process, though perhaps a little startling inasmuch as we assume without proof the inversion of a *compound* functional operation, to show the "flexibility" (as its inventor calls it) of the Calculus. We must now interpret the equation we have arrived at, and will do so in the most elementary way by at once recurring to the $i, j, k$ notation.

[*We may observe, once for all, that objection cannot fairly be taken to the apparent length or tediousness of the solution we are about to give, as compared with the well-known corresponding one in Cartesian Coordinates. It must be remembered that the reader is supposed to be learning the principles of Quaternions by their applications. Had he previously acquired these, as he must have acquired those of Algebra before commencing a Cartesian solution, the quaternion work of any of the problems we have given in illustration need not have exceeded three or four lines, whereas at the present rate of expanded illustration we may require two or three pages. In future papers of this series the enormous shortening of the* *analytical work of a problem by the introduction of quaternions will be seen at a glance, as we shall then feel it right to give quaternions their full scope, the reader being supposed by that time to be pretty well familiarized with the ordinary rules, so that no observation will be requisite unless some perfectly novel rule or consideration be introduced.*]

This premised, we may write

$$\alpha = -iSi\alpha - jSj\alpha - kSk\alpha,$$

which may be verified at once by operating successively by $S.i$, $S.j$, and $S.k$; or may be deduced from (XI.) by putting $\alpha$ for $\delta$, and $i, j, k$ for $\alpha, \beta, \gamma$.

Our equation becomes

$$S.(iSi\alpha + \ldots)(1 - \rho^2\phi)^{-1}(iSi\alpha + \ldots) = 0.$$

Now, by (XXXII.),

$$\phi i = -\frac{i}{a^2},$$

and therefore $$(1 - \rho^2\phi)\, i = \left(1 + \frac{\rho^2}{a^2}\right) i;$$

whence $$(1 - \rho^2\phi)^{-1} i = \frac{a^2 i}{a^2 + \rho^2},$$

with similar expressions in $j$ and $k$. Hence (1) becomes

$$S.(iSi\alpha + \ldots)\left(\frac{a^2 iSi\alpha}{a^2 + \rho^2} + \ldots\right) = 0,$$

or, by the rules of $i, j, k$,

$$\frac{a^2 (Si\alpha)^2}{a^2 + \rho^2} + \frac{b^2 (Sj\alpha)^2}{b^2 + \rho^2} + \frac{c^2 (Sk\alpha)^2}{c^2 + \rho^2} = 0,$$

a *quadratic* in $\rho^2$. Developed, this takes the form

$$\rho^4 \{a^2 (Si\alpha)^2 + b^2 (Sj\alpha)^2 + c^2 (Sk\alpha)^2\}$$
$$+ \rho^2 \{(b^2 + c^2)\, a^2 (Si\alpha)^2 + (c^2 + a^2)\, b^2 (Sj\alpha)^2 + (a^2 + b^2)\, c^2 (Sk\alpha)^2\}$$
$$+ a^2 b^2 c^2 \{(Si\alpha)^2 + (Sj\alpha)^2 + (Sk\alpha)^2\} = 0,$$

or, by (XXXII.),

$$\rho^4 S\alpha\phi^{-1}\alpha + \rho^2 \{(a^2 + b^2 + c^2)\, S\alpha\phi^{-1}\alpha - S\alpha\phi^{-2}\alpha\} - a^2 b^2 c^2 \alpha^2 = 0 \ldots (2),$$

from which the values of $\rho^2$ (the squares of the lengths required, taken negatively) are found. Substituting these in former equations we get the directions. In a subsequent paper we will show how by the properties of $\phi$ discovered, but not yet published, by Sir W. R. Hamilton, all that precedes may be deduced almost intuitively. Meanwhile we remark that the product of the values of $\rho^2$ is

$$-\frac{a^2b^2c^2\alpha^2}{S\alpha\phi^{-1}\alpha},$$

and therefore that the *area of the central section of the ellipsoid perpendicular to* $\alpha$ *is*

$$\frac{\varpi abc T\alpha}{\sqrt{(S\alpha\phi^{-1}\alpha)}},$$

a result which we will find useful. We may on a future occasion show how to deduce this result by quaternion integration.

Meanwhile we see that *the normals to central plane sections of equal area form the cone*

$$S\alpha\phi^{-1}\alpha = C\alpha^2.$$

Also that *the central section perpendicular to* $\alpha$ *is a circle* if the roots of (2) are equal, *i.e.*, if

$$-4a^2b^2c^2\alpha^2 S\alpha\phi^{-1}\alpha = \{(a^2+b^2+c^2)\, S\alpha\phi^{-1}\alpha - S\alpha\phi^{-2}\alpha\}^2 \ldots (3).$$

For variety let us pass to Cartesian Coordinates. The last equation becomes, if $\alpha = ix + jy + kz$,

$$4a^2b^2c^2\,(x^2+y^2+z^2)\,(a^2x^2+b^2y^2+c^2z^2)$$
$$= \{(b^2+c^2)\,a^2x^2 + (c^2+a^2)\,b^2y^2 + (a^2+b^2)\,c^2z^2\}^2,$$

whence $(b^2-c^2)^2\,a^4x^4 + (c^2-a^2)^2\,b^4y^4 + (a^2-b^2)^2\,c^4z^4$

$$-2\,(a^2-b^2)\,(b^2-c^2)c^2a^2z^2x^2 - 2\,(b^2-c^2)\,(c^2-a^2)\,a^2b^2x^2y^2$$
$$-2\,(c^2-a^2)\,(a^2-b^2)\,b^2c^2y^2z^2 = 0,$$

or

$$\{(b^2-c^2)\,a^2x^2 - (a^2-b^2)\,c^2z^2\}^2 - \{2\,(b^2-c^2)\,(c^2-a^2)\,a^2b^2x^2y^2$$
$$+2\,(c^2-a^2)\,(a^2-b^2)\,b^2c^2y^2z^2 - (c^2-a^2)^2\,b^4y^4\} = 0.$$

Now as $a^2 > b^2 > c^2$, *all* the terms inside the last bracket are negative, so that that quantity (if it exist) is positive, and forms a sum equal to zero when added to another essentially positive quantity. Hence both must vanish, and therefore $y = 0$ and

$$(b^2-c^2)\,a^2x^2 - (a^2-b^2)\,c^2z^2 = 0,$$

so that the only values of $\alpha$ which satisfy (3) are parallel to

$$\frac{\sqrt{(a^2-b^2)}}{a}\,i \pm \frac{\sqrt{(b^2-c^2)}}{c}\,k,$$

the known normals to the circular sections. *Why* we have introduced this piece of Algebra will be evident when we attack the same question by quaternions in a future paper.

In the next article we will consider, in the first place, Sir W. R. Hamilton's form of the equation to the ellipsoid referred to its cyclic normals

$$T(\iota\rho + \rho\kappa) = \kappa^2 - \iota^2,$$

for the understanding of which some farther explanations of elementary theorems in quaternions will be required.

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# ON TRIANGULAR AND TETRAHEDRAL COORDINATES.

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1. THE principal object of this paper is to exhibit the advantages gained by using in plane geometry *triangular*, and in solid geometry *tetrahedral*, coordinates.

The coordinates themselves will be taken to represent *the ratios* of the triangles or pyramids, whose vertices are at the proposed point, to the triangle or tetrahedron of reference. This will be found explained in Ferrers' *Trilinear Coordinates*, Chapter V., and in Frost and Wolstenholme's *Solid Geometry*, Art. VIII.

In following out the above plan, we shall of course have to mention some well-known theorems, others however will be given which (to the best of our knowledge) have not appeared elsewhere.

In the examples adduced to illustrate the subject, any results which were originally given in a different form, will be adapted to the systems of coordinates here used.

2. The following summary may be given of the advantages alluded to.

In plane geometry, the equation to the line at infinity being $\alpha+\beta+\gamma=0$, the elimination of one of the variables between this and the equation to the curve is more simple. Also the equations, thence deduced, for the centre of a conic $S=0$, viz., $\frac{dS}{d\alpha}=\frac{dS}{d\beta}=\frac{dS}{d\gamma}$; are more convenient than those in trilinears.

Similar remarks apply to Solid Geometry.

The connexion between triangular and tetrahedral, and the tangential coordinates respectively corresponding to them, is more symmetrical than in the other case; thus the asymptotes of a curve in tangential coordinates are found from $\frac{dS}{d\alpha}=\frac{dS}{d\beta}=\frac{dS}{d\gamma}$.

Professor Slesser's equations (*Quarterly Journal*, Vol. II., p. 357) afford another instance of the simplicity of the results met with in these systems.

Lastly, the theorems in *projections* and *corresponding points*, to which we shall introduce the reader, seem to belong peculiarly to these coordinates.

We proceed to establish the truth of what we have just stated.

3. *The triangular coordinates of a point remain unaltered, when the point and the triangle of reference are projected orthogonally.* For the ratio of areas is a projective property. If therefore a curve and triangle of reference are projected, the equation to the new curve, referred to the projected triangle, is the same as the original one.

Ex. *To find the equation to the ellipse of least area circumscribing the triangle of reference.*

Project the triangle into an equilateral one, the ellipse of minimum area will be the circumscribing circle, whose equation is $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=0$. This therefore will be the equation to the ellipse required.

In this ellipse the tangent at either angular point is parallel to the opposite side, and the centre is the centre of gravity of the triangle.

Similarly the greatest ellipse which can be inscribed in the triangle, has for its equation

$$\sqrt{(\alpha)}+\sqrt{(\beta)}+\sqrt{(\gamma)}=0.$$

The centre of this likewise, is the centre of gravity of the triangle, and its points of contact bisect the sides.

4. *To find the condition that the general equation of the second degree may represent an ellipse, parabola, or hyperbola.*

Let the equation be

$$A\alpha^2+C\beta^2+C\gamma^2+2A'\beta\gamma+2B'\gamma\alpha+2C'\alpha\beta=0,$$

eliminate $\gamma$ between this and $\alpha+\beta+\gamma=0$, therefore

$$A\alpha^2+B\beta^2+C(\alpha+\beta)^2-2(A'\beta+B'\alpha)(\alpha+\beta)+2C'\alpha\beta=0.$$

The points thus determined, where the conic cuts the line at infinity, will be imaginary, real, or coincident according as

$$(C-A'-B'+C')^2-(A+C-2B')(B+C-2A'),$$

*i.e.*,

$$A'^2+B'^2+C'^2-2B'C'-2C'A'-2A'B'-BC-CA-AB$$
$$+2AA'+2BB'+2CC',$$

is negative, zero, or positive; and in these cases respectively, the conic is an ellipse, parabola, or hyperbola.

5. The condition that the above general equation may represent a circle is shown to be

$$\frac{B+C-2A'}{a^2}=\frac{C+A-2B'}{b^2}=\frac{A+B-2C'}{c^2}\ \ldots.(\alpha).$$

*Quarterly Journal*, Vol. II., p. 267.

Now since any ellipse can be orthogonally projected into a circle, we ought to be able to obtain the condition of Art. 4 from these latter equations.

The square of each of the ratios in ($\alpha$) is equal to

$$\frac{4(C-A'-B'+C')^2-4(A+C-2B')(B+C-2A')}{(a^2+b^2-c^2)^2-4a^2b^2}$$

$$=4.\frac{A'^2+B'^2+C'^2-2B'C'-2C'A'-2A'B'-BC-CA-AB+2AA'+2BB'+2CC'}{-(a+b+c)(b+c-a)(c+a-b)(d+b-c)},$$

and the denominator being negative, the numerator must be so also.

**Problem.** *If the radii of the ellipse* $\frac{l}{\alpha}+\frac{m}{\beta}+\frac{n}{\gamma}=0$ *drawn parallel to the sides of the triangle of reference be* $\rho_a, \rho_b, \rho_c$, *then*

$$\frac{l\rho_a^2}{a^2}=\frac{m\rho_b^2}{b^2}=\frac{n\rho_c^2}{c^2}.$$

If the ellipse be projected into a circle of radius $\rho$, circumscribing a triangle whose sides are $a'$, $b'$, $c'$; we have

$$\frac{l}{a'^2}=\frac{m}{b'^2}=\frac{n}{c'^2}.$$

Also, since the ratio of parallel straight lines remains unaltered by projection,

$$\frac{a'^2}{\rho^2}=\frac{a^2}{\rho_a^2},\quad \frac{b'^2}{\rho^2}=\frac{b^2}{\rho_b^2},\quad \frac{c'^2}{\rho^2}=\frac{c^2}{\rho_c^2},$$

therefore

$$\frac{l\rho_a^2}{a^2}=\frac{m\rho_b^2}{b^2}=\frac{n\rho_c^2}{c^2}.$$

Similarly if $R_a$, $R_b$, $R_c$ be the radii parallel to the sides, in the curve represented by the general equation

$$\frac{R_a^2(B+C-2A')}{a^2}=\frac{R_b^2(C+A-2B')}{b^2}=\frac{R_c^2(A+B-2C')}{c^2}.$$

*Having given the radii of a conic parallel to the sides of the triangle of reference to find its area.*

Let it be projected into a circle of radius $r$, referred to a triangle whose sides are $a'$, $b'$, $c'$: also let $S$ be the area of the conic, $\Delta$ that of the triangle of reference, therefore

$$\frac{S^2}{\Delta^2}=\frac{\pi^2r^4}{s'(s'-a')(s'-b')(s'-c')}.$$

But if $\rho_a$, $\rho_b$, $\rho_c$ be the given radii parallel to the sides $a$, $b$, $c$ of the triangle of reference

$$\frac{a'}{r}=\frac{a}{\rho_a},\ \frac{b'}{r}=\frac{b}{\rho_b},\ \frac{c'}{r}=\frac{c}{\rho_c};\ \frac{2S'}{r}=\frac{a}{\rho_a}+\frac{b}{\rho_b}+\frac{c}{\rho_c}=2\Sigma\ (\text{say});$$

therefore
$$\frac{S^2}{\Delta^2}=\frac{\pi^2}{\Sigma\left(\Sigma-\frac{a}{\rho_a}\right)\left(\Sigma-\frac{b}{\rho_b}\right)\left(\Sigma-\frac{c}{\rho_c}\right)}.$$

6. There are two methods of proceeding in the application of projections; either we project the triangle into another given one, and find what becomes of any curve referred to the former, or we project the curve into its corresponding one, and find the projection of the triangle. Suppose now we have *any conic* given, and assume that it is projected into a circle; then we have the equations ($\alpha$) of Art. 5, from which to find the ratio of the sides of the triangle. The subsequent investigation shews us that we should not get a real triangle unless the given conic were an ellipse; nevertheless we should obtain analytical values for the sides; and provided the results deduced from these values were intelligible, it would not be necessary to shew that quantities introduced for a particular purpose, and then *eliminated*, satisfied the condition that any two were together greater than the third, or were all positive, or even real. We feel quite safe therefore in asserting the propositions just proved for ellipses, to be true for any conics. On this dependence on the continuity of the algebraical symbols, see Ferrers' *Trilinear Coordinates*, Ch. VIII., Art. 18.

7. In SOLID GEOMETRY tetrahedral coordinates occupy the position which in plane geometry is held by triangulars: and there is a close connexion between the two, for we observe at once that the tetrahedral coordinates of any point in one of the faces of the tetrahedron of reference, are the same as its triangular coordinates referred to that face. If therefore we want the equation in triangular coordinates to a section by one of the faces of the tetrahedron, of a

surface whose equation is given we have merely to put one of the variables equal to zero.

**Example.** The equation to the circle circumscribing a triangle whose sides are $a$, $b$, $c$ is $\Sigma(\alpha^2\beta\gamma)=0$; hence the equation to the sphere circumscribing the tetrahedron of reference $ABCD$ is $\Sigma(\overline{AB}^2\alpha\beta)=0$.

For the equation to each section by one of the faces of $ABCD$ must be of the form given above for a circle.

8. *Having given the equation to any surface in tetrahedral, to find its equation in tangential coordinates.*

Let $\phi(\alpha\beta\gamma\delta)=0$ be the equation to any surface

$$\alpha'\frac{d\phi}{d\alpha}+\beta'\frac{d\phi}{d\beta}+\gamma'\frac{d\phi}{d\gamma}+\delta'\frac{d\phi}{d\delta}=0,$$

the equation to the tangent plane at the point $\alpha$, $\beta$, $\gamma$, $\delta$; its coordinates in the tangential system are proportional to $\frac{d\phi}{d\alpha}$, $\frac{d\phi}{d\beta}$, $\frac{d\phi}{d\gamma}$, $\frac{d\phi}{d\delta}$, (Frost and Wolstenholme's *Geometry*, Art. 97).

Hence, if we eliminate $\alpha$, $\beta$, $\gamma$, $\delta$ between the given equation to the surface and the equations

$$\frac{\frac{d\phi}{d\alpha}}{\alpha'}=\frac{\frac{d\phi}{d\beta}}{\beta'}=\frac{\frac{d\phi}{d\gamma}}{\gamma'}+\frac{\frac{d\phi}{d\delta}}{\delta'},$$

the result will be the equation required.

Ex. Suppose, for the sake of distinction, that the accented letters refer to the tangential system, the others to the tetrahedral, then the following pairs of equations refer to the *same* surfaces,

$$\left.\begin{array}{l}\sqrt{(l\alpha)}+\sqrt{(m\beta)}+\sqrt{(n\gamma)}+\sqrt{(r\delta)}=0\\ \dfrac{l}{\alpha'}+\dfrac{m}{\beta'}+\dfrac{n}{\gamma'}+\dfrac{r}{\delta'}=0\end{array}\right\},$$

$$\left.\begin{array}{l}\sqrt{(l\alpha')}+\sqrt{(m\beta')}+\sqrt{(n\gamma')}+\sqrt{(r\delta')}=0\\ \dfrac{l}{\alpha}+\dfrac{m}{\beta}+\dfrac{n}{\gamma}+\dfrac{r}{\delta}=0\end{array}\right\}.$$

From the fundamental definitions the tangential equations represent the conditions that the plane $\alpha'\alpha+\beta'\beta+\ldots=0$, should touch the surfaces represented by the corresponding tetrahedral coordinates. Similarly the tetrahedral equations represent the conditions that the point $\alpha\alpha'+\ldots=0$ should lie on the surface represented by the tangential equation.

9. Let us next take two systems of coordinates, referred respectively to different tetrahedra. We shall call points in the two systems which have the *same coordinates* CORRESPONDING POINTS.

Evidently then a straight line in one system *corresponds* to a straight line in the other; a plane to a plane, &c.; and any surface in one system has the *same equation* as the corresponding surface in the other.

*The plane at infinity corresponds to itself.*

*Parallel planes correspond to parallel planes.* For their intersection is a line on the plane at infinity.

*Parallel lines correspond to parallel lines.* For a pair of parallel lines are in the same plane and meet at infinity: or they may be considered as the intersections of two pairs of parallel planes.

10. Corresponding surfaces having the *same equation*, their tangent and polar planes belonging to corresponding points, themselves correspond, as do also the poles of corresponding planes, and hence the centres of corresponding surfaces.

*The volumes of corresponding surfaces are in the ratio of their respective tetrahedra.*

Professor Slesser has shewn (*Quarterly Journal*, Vol. II., p. 357) that the volume of a surface $=6V\iiint d\alpha d\beta d\gamma$, $V$ being the volume of the tetrahedron of reference; and the equations to the surfaces from which the limits are obtained, being *the same*, the volumes of the surfaces vary as those of the tetrahedra.

*The centres of gravity of corresponding surfaces, themselves correspond.*

A plane dividing the surface into two parts of equal volume, passes through the centre of gravity: and the same is true for the corresponding plane with respect to the corresponding surface. The intersections of the two such systems of planes determine the two centres of gravity, which therefore correspond.

*The ratio of parallel straight lines is the same as that of the corresponding lines.*

The extremities of equal and parallel straight lines being joined form a parallelogram, to which corresponds a parallelogram, hence equal and parallel straight lines, correspond to others equal and parallel.

Hence, considering any two parallel straight lines as multiples of two small *equal* and parallel lines, we deduce the proposition above stated.

11. We have thus obtained results analogous to the fundamental projective properties in plane geometry; in fact the definition we have given of corresponding points, embraces projections in plane geometry, as may be seen by comparing Art. 3 with what was said at the commencement of Art. 7.

12. *If two surfaces are similar, and similarly situated, their corresponding surfaces are also similar, and similarly situated.*

For if lines drawn from a fixed point, to meet one surface, bear a constant ratio to those drawn parallel to them, from another fixed point, to meet another surface; so will they also from the corresponding points, to the corresponding surfaces.

*To find the conditions that two surfaces represented by the general equations of the second degree may be similar and similarly situated.*

Adopting the notation of the *Quarterly Journal*, Vol. II., p. 267, let the equations to the two surfaces be

$$\Sigma(A\alpha^2) + 2\Sigma([ab]\,\alpha\beta) = 0, \text{ and } \Sigma(A'\alpha^2) + 2\Sigma([a'b']\,\alpha\beta) = 0.$$

If a sphere correspond to one of these, another sphere must correspond to the other; let $\overline{AB}$, &c. be the lengths of the edges of the corresponding tetrahedron, then we have

$$\frac{A+B-2\,[ab]}{\overline{AB}^2}=\&\text{c.}$$

Similarly also
$$\frac{A'+B'-2\,[a'b']}{\overline{AB}^2}=\&\text{c.}$$

Therefore
$$\frac{A+B-2\,[ab]}{A'+B'-2\,[a'b']}=\&\text{c.},$$

which are the required conditions of similarity.

13. **Prob. I.** *To find the equation of the ellipsoid of minimum volume, circumscribed about the tetrahedron of reference.*

Take as a second tetrahedron, a regular one; the ellipsoid of minimum volume for this is obviously a sphere whose equation is

$$\alpha\beta+\beta\gamma+\gamma\alpha+\alpha\delta+\beta\delta+\gamma\delta=0.$$

This is therefore the equation to the ellipsoid of minimum volume circumscribed about the given tetrahedron.

In this ellipsoid the tangent planes at the angular points are parallel to the opposite faces, and the centre of the surface is the centre of gravity of the tetrahedron. It is only necessary to observe that this is true in the case of the sphere and the regular tetrahedron.

**Prob. II.** *To find the equation to the ellipsoid of greatest volume inscribed in the tetrahedron of reference.*

As before, the equation to the sphere inscribed in a regular tetrahedron is

$$\alpha^2+\beta^2+\gamma^2+\delta^2-\alpha\beta-\beta\gamma-\gamma\alpha-\alpha\delta-\beta\delta-\gamma\delta=0,$$

which is the equation required.

The centre of this ellipsoid is also the centre of gravity of the tetrahedron and it touches each face in its centre of gravity.

**Prob. III.** *If $R_{ab}$ be the radius of the surface*

$$\Sigma\{A\alpha^2 + 2[ab]\alpha\beta\} = 0,$$

*parallel to the edge AB of the tetrahedron of reference*, then

$$\frac{R_{ab}^2\{A+B-2[ab]\}}{\overline{AB}^2} = \frac{R_{bc}^2\{B+C-2[bc]\}}{\overline{BC}^2} = \&c.$$

For if $A'B'C'D'$ be the corresponding tetrahedron when the corresponding surface become a sphere of radius $\rho$,

$$\frac{A+B-2[ab]}{\overline{A'B'}^2} = \&c.$$

And
$$\frac{\rho}{\overline{A'B'}} = \frac{R_{ab}}{\overline{AB}}, \&c.,$$

whence the proposition follows.

**Prob. IV.** The equation to the circumscribing conicoid whose radii parallel to the edges of the tetrahedron are respectively $R_{ab}$, &c., is

$$\Sigma\left(\frac{\overline{AB}^2}{R_{ab}^2}\alpha\beta\right) = 0.$$

This follows in a manner similar to the above, from the equation to the circumscribing sphere.

14. By taking a regular tetrahedron to correspond to a given one, propositions respecting volumes, parallel lines and planes frequently become self-evident. The reader may refer to the article on Tetrahedrons at page 156 of this volume.

We will only further suggest to any one who may wish to pursue the subject, that the relations of some other lines and surfaces, to a given one, remain unaltered in the corresponding system of surfaces; as, for example, the generating lines, and asymptotic cones of hyperboloids.

# CONSERVATION OF POTENCY.

By *J. M. Wilson, M.A.*, Fellow of St. John's College, Cambridge, and Natural Philosophy Master in Rugby School.

(*Continued from page* 22.)

VARIOUS criticisms have reached me on the new term 'Conservation of potency' that I have ventured to suggest. One, from an eminent experimentalist is as decidedly favourable, as another from a valued correspondent and writer in this *Journal* is the reverse. He writes, 'You have done well in your paper: but confound potency, pottering idea: the *word* and the *thing* is ENERGY.' Potential energy expresses the required conception, and is an exact equivalent for potency; potentiality would be still better, but is sesquipedalian. It is the translation of δύναμις and not of ἐνεργεία that is wanted.

I shall endeavour in the present number to shew by what kind of reasoning the grand theorem of conservation of potency has been extended from such cases of dynamics of a particle, or of rigid bodies as were spoken of above, until it includes or bids fair to include every manifestation of what is termed force. And in so doing, I shall endeavour not to forget that this is the 'Journal of the Junior Students.'

If an intelligent mechanic were invited to define what he meant by the term force, he would probably give an answer, which though not so accurately expressed, would be equivalent to the ordinary definition. Force is that which produces or tends to produce motion. A little more acquaintance with the ordinary language of science would make this definition seem very inadequate. What for example would be understood by the chemical or vital force? If one attempt to alter the definition so as to include these and other manifestations of what the physicist terms force, it loses clearness as it gains comprehensiveness, and finally includes everything and means nothing. To define is to consent to be unintelligible.

In fact as the definition stands, it is essentially a mathematical and not a physical definition. It is the definition of a mode of estimating force, by which force and its effects are brought under the domain of mathematical symbols, and not of force itself: thus conforming to the only correct definition of mathematics as the science which effects the *indirect* measurement of magnitudes by the precise relations which exist among them. And motion being by far the most measurable effect of force, it is seized upon by mathematicians as the basis of their calculations, and that not only in dynamics proper, but in all physico-mathematical investigations; and hence we arrive at the theorem, that *when the whole of the results of any force can be directly or indirectly contemplated as producing motion, the science is theoretically completely in the grasp of mathematics.*

Now the most remarkable fact in physics is the correlation of forces. Heat, chemical action, electricity in its different forms seem to imply one another, as convertible manifestations of one and the same force. Consider a common experiment—the giving galvanic shocks from a small coil. There we have chemical action in the cells, heat and a current of electricity in the wires, sparks of light at the contact breaker, sound and motion in the spring keeper, and magnetism in the electro-magnet, and peculiar nervous action in the hand. There must be some common basis, some common measure, of these manifestations of force, and what can this be but motion? The mind can distinctly conceive of no physical action except matter in motion. And though this is no proof that the theory is true in fact, it is a strong reason for its adoption. An additional confirmation is derived from the consideration, that these forces are necessarily estimated by motion, as heat by the expansion it produces; electricity by its producing a couple, or by electro-chemical decomposition which is volumetric.

When this fact of correlation was once distinctly seen; other facts began to assume a new importance. You can make heat produce electricity, and electricity produce heat;

can you not then make a perpetual source of heat? Will not the heat produced by the electricity be more than enough to keep up the requisite supply of electricity? In whatever form this question is put, the answer is invariably no. So that perpetual motion can be looked for from combinations of magnets, with no more hope of success than from balls in a basin or from pendulums and springs. These facts led up to the grand theorem of conservation of force, viz. that force cannot be created; and this was soon extended to the additional conclusion that force cannot be destroyed. Just as matter can be neither made nor unmade; so it is with force. This is the real limit to man's power over nature.

Faraday's discovery of the definite nature of electrolysis led more directly to this theorem than any other investigation, so far only however as concerned the quantitative convertibility of chemical and electrical action. His further discovery that electrical induction is a molecular transmission of force in definite lines, and of the existence of magnetic lines of force, recalled attention to the most favourite speculation of previous ages, the molecular composition of matter. Perhaps nothing has been the subject of more profound enquiry, than the internal structure of bodies. And this present theorem of the conservation of force, forms at once a guide and a test to speculation on this subject.

Perhaps it will not be out of place to indicate some of the phenomena, which, in the history of science, have been considered the great arguments for a definite molecular structure of bodies, and serve also to give more precise indications of its nature.

Nearly all bodies with which we are acquainted, are capable of assuming the three forms of solid, liquid and gaseous, under the influence of heat. It was tempting to suppose that such bodies consisted of molecules which attract one another at a certain distance (and form a solid); are neutral at a slightly increased distance (a fluid), and repel one another when the distance between is still further increased by increasing the heat, and form an elastic gas. But such rough physical speculations, are little more than the guesses of the ancient Greeks; and would have remained unnoticed, had they not been confirmed by Dalton's Atomic Theory. From this it appears, that the numerical laws of the composition of compound bodies may be deduced from the single assumption, that bodies consist of molecules of *definite weight.* The confirmation this has received by the study of *atomic volumes* is most important. It may some day be the definition of chemistry, that it is the science that deals with the mutual actions of the atoms of bodies. It is difficult to convey to any one ignorant of chemistry, any conception of what the atomic theory is in that science. It may be compared to the invention of coordinates, and the expressing curves by an equation, in the history of mathematics: the sole guide through endless perplexities.

Definite crystalline form seems to offer another clue to molecular structure; but it is one which has at present led to no well marked results; if we except the remark of Sir John Herschel, recently confirmed and extended by M. Pasteur's brilliant discoveries on the connection between the hemihedrity of crystals, and the direction of the deviation of the plane of polarization which is given by their solutions.

All the phenomena of isomorphism, of homologous series in organic chemistry, and all the more recent results obtained in the science of heat, point the same way. It is pre-eminently the distinguishing feature of the philosophy of our time, that it is unravelling the microcosm.

Besides, and it is on this account that it is treated of at all in this magazine, it is the only basis for mathematical calculation. And for mathematicians it possesses extraordinary attractions. Various results of great interest have been attained on strictly mathematical hypotheses; such as some by Maxwell and Challis; but no hypothesis has yet been made, which has the requisite universality. What should be here pointed out is, the immediate relation these hypotheses bear to the observed facts of correlation of forces, and the conservation of potency. If the forces of heat and electricity consist in molecular motion, it is intelligible how they may produce one another; and the correlation ceases to be a mystery: but still further, the conservation of force *follows as a mathematical deduction* independent of experiment. This was proved in my first paper; and conversely, that if experiment verifies this theorem, then the mathematical expression of it is identical with the expression of molecules attracting one another.

And experiment has verified it, in the hands of Joule and Thomson; and thus a new era in physical science will be inaugurated, when determinate numerical relations are made to bind together all the various physical actions, an era in physical science corresponding to the discovery of Dalton's laws of chemical combination.

It would require too great length, and be foreign to the objects of this Magazine to introduce any account of the experimental confirmation of the theorem. But some of the consequences of this theorem may perhaps be noticed. Thomson has republished in *Macmillan's Magazine* for March, 1862, a very interesting paper on the age of the Sun's heat, in which this theorem is assumed of course throughout, and its consequences traced with a lucidity of exposition that cannot be too much admired. The Sun is then spending his force on us, and it is being dissipated. But consider what we owe, not only to his present, but to his past expenditure. Under his influence, and *absorbing his heat*, the leaves of the trees of the carboniferous forests, millions of years ago decomposed the carbonic acid, and built themselves up into the hydrocarbons which now form coal. In that form, by the isolation of hydrogen and carbon from the oxygen with which they were combined, his heat has been stored for us, and *that heat* is given out in our engine fires, and converted by the agency of the violent molecular action of steam at a high temperature into pressure and mechanical force. We are not living on our income of heat from the Sun, but consuming his buried hoards.

So the application of this theorem to organic life and physiology, is a strange mixture of romance with fact.

Perhaps these few remarks may serve to widen the range of facts included in the theorem of conservation of *vis viva* as proved in our text books, and to place our mathematics in their true position as the handmaids and the language of physical philosophy.

NOTE.—Since the above was in type, an article on this subject has appeared in *Macmillan's Magazine.* It will be a useful exercise for any *young* student to read it and note its errors and fallacies.

It is much to be regretted, that a Magazine so well conducted should have admitted such a production.

## ON CURVATURE.

THE similarity of the formulæ for the radii of principal curvature, and of torsion in curves of double curvature, and other expressions respecting the curvature of curves or surfaces, naturally suggests that they all depend upon the same general principle, or are particular cases of the same proposition.

The following propositions include all the cases usually discussed by the student.

1. $OP$, $OQ$ are two straight lines *meeting in* $O$, at an indefinitely small angle $d\theta$. If $l$, $m$, $n$ be the direction-cosines of $OP$, referred to any rectangular axes; $l+dl$, $m+dm$, $n+dn$ the direction-cosines of $OQ$, then

$$\left(\frac{dl}{d\theta}\right)^2 + \left(\frac{dm}{d\theta}\right)^2 + \left(\frac{dn}{d\theta}\right)^2 = 1 \text{ ............ } (A).$$

And if $PQ$ $(=ds)$ be the small arc of any curve in the same plane as $OP$, $OQ$, and cutting these lines at right angles, the direction-cosines of the tangent to $PQ$, are

$$\frac{dl}{d\theta},\ \frac{dm}{d\theta},\ \frac{dn}{d\theta} \text{ .................... } (B).$$

Or if $OP=r$, then since $r=\frac{ds}{d\theta}$, the direction-cosines of $PQ$ are

$$r\frac{dl}{ds},\ r\frac{dm}{ds},\ r\frac{dn}{ds}\ \ldots\ldots\ldots\ldots\ (B'),$$

and we have

$$\frac{1}{r^2}=\left(\frac{dl}{ds}\right)^2+\left(\frac{dm}{ds}\right)^2+\left(\frac{dn}{ds}\right)^2\ \ldots\ldots\ldots\ (A').$$

We shall speak of $PQ$ as the trajectory of the lines $OP$, $OQ$.

Consider $P$, $Q$ as points on a sphere whose centre is $O$ and radius $r$, and let $x$, $y$, $z$ be the coordinates of $P$; $x+dx$, $y+dy$, $z+dz$ those of $Q$, therefore

$$x=lr.$$

Differentiating $\frac{dx}{ds}=r\frac{dl}{ds}$

Similarly $\frac{dy}{ds}=r\frac{dm}{ds}$

and $\frac{dz}{ds}=r\frac{dn}{ds}$

$$\left.\begin{aligned}\frac{dx}{ds}&=r\frac{dl}{ds}\\ \frac{dy}{ds}&=r\frac{dm}{ds}\\ \frac{dz}{ds}&=r\frac{dn}{ds}\end{aligned}\right\}.$$

Squaring and adding $\frac{1}{r^2}=\left(\frac{dl}{ds}\right)^2+\left(\frac{dm}{ds}\right)^2+\left(\frac{dn}{ds}\right)^2.$

Again, since $r=\frac{ds}{d\theta}$, the above equations may be written

$$\left.\begin{aligned}\frac{dx}{ds}&=\frac{dl}{d\theta}\\ \frac{dy}{ds}&=\frac{dm}{d\theta}\\ \frac{dz}{ds}&=\frac{dn}{d\theta}\end{aligned}\right\},$$

$$1=\left(\frac{dl}{d\theta}\right)^2+\left(\frac{dm}{d\theta}\right)^2+\left(\frac{dn}{d\theta}\right)^2.$$

### APPLICATION TO THE CURVATURE OF CURVES IN SPACE.

2. The accompanying figure 36 is similar to that used in the solutions to the Senate-House Problems, 1860; for a more elaborate one, we may refer the reader to any of the standard works on Solid Geometry.

$PP'$, $QQ'$ are consecutive tangents to the same curve, which may be considered to lie in the osculating plane of the curve at $P$, or more correctly at some point between $P$ and $Q$; the trajectory of these lines is therefore parallel to the radius of curvature $PC$ at $P$.

$PM$, $QM$ are tangents to any one of the evolutes of the curve, *i.e.* any two normals at $P$ and $Q$ which meet one another; their trajectory is of course $PQ$.

$OC$, $OC'$ are consecutive tangents to the arête de rebroussement or locus of centres of spherical curvature, they are normals to the osculating planes at $P$ and $Q$, and the angle between them is therefore what is called the angle of torsion, their trajectory is $PC$, the radius of curvature at $P$.

$OC$ is not the radius of torsion (this latter being only a convenient name for the ratio of $PQ$ to the angle of torsion), but bears the same ratio to it as $CC'$ does to $PQ$.

Let $PC = \rho$, $PQ = ds$, the angle between $PP'$, $QQ' = d\phi$, that between $OC$, $OC' = du$, and the radius of torsion $= r$,

therefore $$\rho = \frac{ds}{d\phi}, \quad r = \frac{ds}{du},$$

and let $\alpha\beta\gamma$ be the angles the principal normal $PC$ makes with the axis.

PROP. I. If $l, m, n$ are the direction-cosines of $PM$, then $PM\,\dfrac{dl}{ds}$, &c. are those of the tangent $PP'$, and

$$\left(\frac{1}{Pm}\right)^2 = \left(\frac{dl}{ds}\right)^2 + \left(\frac{dm}{ds}\right)^2 + \left(\frac{dn}{ds}\right)^2.$$

The locus of the centres of curvature not being an evolute, the above equation would not be true were $\rho$ written for $PM$ and $l, m, n$ taken as the direction-cosines of $\rho$.

PROP. II. If $l, m, n$ give the direction of the tangent $PP'$, the direction-cosines of the principal normal $Pl$, are

$$\frac{dl}{d\phi}, \quad \frac{dm}{d\phi}, \quad \frac{dn}{d\phi}; \quad \rho\frac{dl}{ds}, \quad \rho\frac{dm}{ds}, \quad \rho\frac{dn}{ds},$$

or, as they are perhaps more generally written,

$$\rho\frac{d^2x}{ds^2} = \cos\alpha, \quad \rho\frac{d^2y}{ds^2} = \cos\beta, \quad \rho\frac{d^2z}{ds^2} = \cos\gamma,$$

also $$\frac{1}{\rho^2} = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2.$$

Prop. III. If $l, m, n$ refer to the normal to the osculating plane, then

$$\cos\alpha = \frac{dl}{du}, \text{ or } r\frac{dl}{ds},$$

$$\cos\beta = \frac{dm}{du}, \text{ or } r\frac{dm}{ds},$$

$$\cos\gamma = \frac{dn}{du}, \text{ or } r\frac{dn}{ds},$$

also
$$\frac{1}{r^2} = \left(\frac{dl}{ds}\right)^2 + \left(\frac{dm}{ds}\right)^2 + \left(\frac{dn}{ds}\right)^2.$$

## FURTHER APPLICATION TO CURVES TRACED UPON SURFACES.

3. We shall here use $\lambda, \mu, \nu$ for the direction-cosines of the *normal to the surface*; $l, m, n$ those of the tangent to a curve on the surface.

(1) If the curve be a *line of curvature* along which consecutive normals to the surface intersect, we have the equations $\rho\frac{d\lambda}{ds} = l$, $\rho\frac{d\mu}{ds} = m$, $\rho\frac{d\nu}{ds} = n$ from Prop. I. The equations of Prop. II. would have given us the principal radius of curvature of the line of curvature, drawn in its osculating plane, which coincides neither in direction nor magnitude with that to the surface.

(2) If the curve be a *geodesic*, since the osculating plane contains the normal to the surface, which is therefore the principal normal of the curve; we have the equations

$$\rho\frac{dl}{ds} = \lambda, \quad \rho\frac{dm}{ds} = \mu, \quad \rho\frac{dn}{ds} = \nu,$$

or according to the more general notation,

$$\frac{\rho\dfrac{d^2x}{ds^2}}{\dfrac{d\phi}{dx}} = \frac{\rho\dfrac{d^2y}{ds^2}}{\dfrac{d\phi}{dy}} = \frac{\rho\dfrac{d^2z}{ds^2}}{\dfrac{d\phi}{dz}},$$

$\phi(x, y, z) = 0$ being the equation to the surface.

4. *As we proceed along one line of curvature, to measure the consequent change in the direction of the tangent to the perpendicular line of curvature.*

Let $(l, m, n)$ $(l', m', n')$ be the tangents to the two lines of curvature, $ds$ an element of the arc of the first. The tangents to the perpendicular lines of curvature at the extremities of the element $ds$ intersect, since they are consecutive normals to another surface meeting the former every where at right angles, along a line of curvature, by Dupin's theorem. Hence, by the proposition at the commencement of this paper, $\rho'$ being a principal radius of curvature of this second surface,

$$l = \rho' \frac{dl'}{ds},$$

$$m = \rho' \frac{dm'}{ds},$$

$$n = \rho' \frac{dn'}{ds}.$$

COR. By means of the last result we may obtain a short proof of the well known property, that along a line of curvature $pD =$ constant.

For since the lines of curvature are parallel to the axes of the section of the ellipsoid by a plane parallel to the tangent plane

$$\frac{ll'}{a^2} + \frac{mm'}{b^2} + \frac{nn'}{c^2} = 0,$$

therefore $$\frac{l' \dfrac{dl'}{ds}}{a^2} + \frac{m' \dfrac{dm'}{ds}}{b^2} + \frac{n' \dfrac{dn'}{ds}}{c^2} = 0,$$

$$\frac{d}{ds}\left(\frac{1}{D'^2}\right) = 0,$$

$$D' = \text{constant},$$

therefore $$pD = \frac{abc}{D'} = \text{constant}.$$

5. To prove $\frac{d\lambda}{dx}+\frac{d\mu}{dy}+\frac{d\nu}{dz}=\frac{1}{\rho_1}+\frac{1}{\rho_2}$.

Let us take as another system of axes $\xi$, $\eta$, $\zeta$ the tangents to the lines of curvature, and the normal at a proposed point; and let their direction be given as before by $(l, m, n)$, $(l', m', n')$, $(\lambda, \mu, \nu)$ respectively. Then at the given point

$$\frac{d\lambda}{dx}=\frac{d\lambda}{d\xi}\frac{d\xi}{dx}+\frac{d\lambda}{d\eta}\frac{dx}{dx}+\frac{d\lambda}{d\zeta}\frac{d\zeta}{dx}.$$

But $\frac{d\lambda}{d\zeta}=0$ the axis of $\zeta$ being the normal; $\frac{d\lambda}{d\xi}$, $\frac{d\lambda}{d\eta}$ are equal to $\frac{l}{\rho_1}$, $\frac{l'}{\rho_2}$ (Art. 3); $\frac{d\xi}{dx}=l$, $\frac{d\eta}{dx}=l'$; therefore

$$\frac{d\lambda}{dx}=\frac{l^2}{\rho_1}+\frac{l'^2}{\rho_2}.$$

Similarly

$$\frac{d\mu}{dy}=\frac{m^2}{\rho_1}+\frac{m'^2}{\rho_2},$$

$$\frac{d\nu}{dz}=\frac{n^2}{\rho_1}+\frac{n'^2}{\rho_2}.$$

Hence

$$\frac{d\lambda}{dx}+\frac{d\mu}{dy}+\frac{d\nu}{dz}=\frac{1}{\rho_1}+\frac{1}{\rho_2}.$$

6. The proof of the proposition at the head of this paper will be seen to depend mainly upon the fact that *OP, OQ meet in O;* that is, that the shortest distance between the two lines is indefinitely small compared to the small quantity *PQ* retained in the investigation. It will not be difficult to shew conversely that if the relations (*A*) and (*B*) of the original proposition hold, the lines do meet according to the above definition.

For let *MN* be the shortest distance between the lines *MP*, *NQ*, and draw *MQ'* parallel to *NQ*, and *PQ'* the trajectory of *MP*, *MP'*; then the formulæ in (*A*) give the direction-cosines of *PQ'* the same as those of *PQ*, which clearly can only be the case when *MN* is indefinitely small compared with *PQ*, or when the straight lines *MP*, *NQ* meet one another.

# ON LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER.

By *James Cockle, M.A.*, of Trinity College, Cambridge.

(*Concluded from page* 173.)

46. SINCE, by (21),

$$q^2 + \frac{dq}{dx} - r = s = \lambda^2 + \frac{d\lambda}{dx} - \mu \text{............(21)},$$

we see that the characteristic $s$ remains unchanged after the substitution of $uv$ for $y$. Moreover when (5) or (8) is soluble all equations having the same characteristic are soluble. Hence if (16) be soluble, then

$$\frac{d^2y}{dx^2} + 2\lambda\frac{dy}{dx} + \left(\frac{d\lambda}{dx} + \lambda^2 - s\right) y = 0 \text{.........(61)},$$

is soluble. Thus, (16) being soluble and $\lambda$ arbitrary, put

$$\lambda = q + a,$$

then

$$\frac{d^2y}{dx^2} + 2(q + a)\frac{dy}{dx} + (r + 2aq + a^2)\, y = 0 \text{......(62)},$$

is soluble.

47. Let $a$, $b$, and $c$ be constants, and $X$ any function whatever of $x$. Then

$$(b + cx)^2\frac{d^2y}{dx^2} + (a + c + X)(b + cx)\frac{dy}{dx} + aXy = 0 \text{...(63)},$$

is soluble; for it is symbolically decomposable into

$$\left\{(b + cx)\frac{d}{dx} + X\right\}\left\{(b + cx)\frac{d}{dx} + a\right\} y = 0 \text{......(64)}.$$

When $X$ is constant (63) becomes a well known soluble form, usually solved by a change of the independent variable.

48. When a change of the independent variable transforms a given equation into another whereof the coefficients are constant, then the given equation is susceptible of a symbolical décomposition in which the order of the symbolical factors is indifferent.

49. When (1) is of the form

$$(\alpha+\beta x+\gamma x^2)\frac{d^2y}{dx^2}+Q\frac{dy}{dx}+Ry=0 \text{.........}(65),$$

it may in general, $\alpha$ and $\gamma$ not vanishing, be transformed into one of the following forms:

$$x^2\frac{d^2y}{dx^2}+Q\frac{dy}{dx}+Ry=0 \text{...............}(66),$$

or

$$(1-x^2)\frac{d^2y}{dx^2}+Q\frac{dy}{dx}+Ry=0 \text{............}(67).$$

50. The following is a soluble case of the latter. Let

$$(1-x^2)\frac{d^2y}{dx^2}+2(K+Hx)\frac{dy}{dx}+(K^2-H^2+K-H)y=0 \text{...}(68).$$

The symbolical decomposition of this equation is

$$\left\{(1+x)\frac{d}{dx}+K-H\right\}\left\{(1-x)\frac{d}{dx}+K+H+1\right\}y=0 \text{...}(69),$$

and the change of $x$ into $-x$ will afford corresponding soluble cases.

51. In (68) let $K=0$ and $H=-\frac{1}{2}$, then we see that

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+\frac{1}{4}y=0 \text{............}(70),$$

admits of the symbolical decomposition

$$\left\{(1+x)\frac{d}{dx}+\frac{1}{2}\right\}\left\{(1-x)\frac{d}{dx}+\frac{1}{2}\right\}y=0 \text{.......}(71).$$

This is a particular case of (15).

52. The Boolian form of this equation is

$$y-\frac{(D-\frac{3}{2})(D-\frac{3}{2}-1)}{D(D-1)}e^{2\theta}y=0 \text{...........}(72).$$

53. Reverting to Art. 15, the solutions of (24) and (26) are, respectively,

$$v=\frac{1}{2(x+c)},\quad v=\frac{-1}{2(x+c)},$$

and the solutions of the equations noticed in Arts. 39–42 are of the form

$$u = \tfrac{1}{2}\Sigma\left(\frac{1}{x+C}\right).$$

54. In (5) let $s = -\lambda^2 - a$, then, proceeding as in Art. 15, we are led to

$$2\lambda^2 - \frac{d\lambda}{dx} + a = 0,$$

or

$$2\frac{d\lambda}{dx} = (2\lambda)^2 + 2a.$$

Hence,

$$\frac{\frac{d(2\lambda)}{dx}}{2a + (2\lambda)^2} = \frac{1}{\sqrt{(2a)}} \cdot \frac{\frac{1}{\sqrt{(2a)}} \cdot \frac{d(2\lambda)}{dx}}{1 + \left(\frac{2\lambda}{\sqrt{(2a)}}\right)^2} = 1,$$

and therefore,

$$\frac{1}{\sqrt{(2a)}} \tan^{-1} \frac{2\lambda}{\sqrt{(2a)}} = x + C,$$

$$\lambda = \tfrac{1}{2}\sqrt{(2a)} \tan\sqrt{(2a)}\,(x+C).$$

Consequently, when

$$s = -\frac{a}{2}\{\tan\sqrt{(2a)}\,(x+C)\}^2 - a \;\ldots\ldots\ldots\ldots(73),$$

(5) is soluble.

55. So, if

$$s = -\lambda^2 + ax^m \ldots\ldots\ldots\ldots\ldots\ldots\ldots(74),$$

where

$$m = \frac{-4n}{2n \mp 1},$$

(5) is soluble, $n$ being an integer.

56. Given the quadratic

$$y^2 + by + cx = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(75),$$

we deduce, successively,

$$2y\frac{dy}{dx} + b\frac{dy}{dx} + c = 0,$$

$$\frac{dy}{dx} = -\frac{c}{2y+b} = \frac{c(2y+b)}{4cx - b^2} = \frac{c}{4cx - b^2}\left\{b - \frac{2}{b}(cx + y^2)\right\},$$

or $$\frac{dy}{dt} = \left\{b - \frac{2}{b}(cx + y^2)\right\} \frac{c}{4cx - b^2} \frac{dx}{dt}.$$

Assume $$\frac{2}{b} \frac{c}{4cx - b^2} \frac{dx}{dt} = 1,$$

or $$\frac{1}{2b} \frac{4c}{4cx - b^2} \frac{dx}{dt} = 1,$$

then $$\frac{1}{2b} \log(4cx - b^2) = t + C,$$

$$4cx - b^2 = e^{2b(t+C)},$$

$$x = \frac{1}{4c}\{e^{2b(t+C)} + b^2\};$$

and, consequently,

$$\frac{dy}{dt} + y^2 = \frac{b^2}{2} - \tfrac{1}{4}\{e^{2b(t+C)} + b^2\} = \tfrac{1}{4}\{b^2 - e^{2b(t+C)}\} \text{...(76)}.$$

Hence the differential equation (76) is connected with the algebraic equation (75).

57. In the system of Art. 31 let $\rho$ be replaced by $\lambda\rho^m$, and let

$$T = 1, \quad U = -\frac{n}{\rho} \frac{d\rho}{dx}.$$

Then we have

$$t = P\lambda\rho^m \text{ .............(77)},$$

$$\frac{dt}{dx} - t \frac{n}{\rho} \frac{d\rho}{dx} + u = Q\lambda\rho^m \text{ .............(78)},$$

$$\frac{du}{dx} - u \frac{n}{\rho} \frac{d\rho}{dx} = R\lambda\rho^m \text{ .............(79)}.$$

58. From (79) we deduce

$$u = \rho^n \int R\lambda\rho^{m-n} dx,$$

and (78) becomes, on substituting for $u$ and $t$, and dividing by $\rho^n$,

$$\frac{1}{\rho^n} \frac{d(P\lambda\rho^m)}{dx} - nP\lambda\rho^{m-n-1}\frac{d\rho}{dx} + \int R\lambda\rho^{m-n} dx = Q\lambda\rho^{m-n},$$

or

$$\frac{d(P\lambda)}{dx} \rho^{m-n} + (m-n) P\lambda\rho^{m-n-1}\frac{d\rho}{dx} + \int R\lambda\rho^{m-n} dx - Q\lambda\rho^{m-n} = 0$$

$$\text{.................(80)}.$$

59. Let $m-n=0$, we have

$$\frac{d(P\lambda)}{dx}+\int R\lambda dx-Q\lambda=0,$$

or, developing, and differentiating again, so as to eliminate the symbol of integration,

$$P\frac{d^2\lambda}{dx^2}+\left(2\frac{dP}{dx}-Q\right)\frac{d\lambda}{dx}+\left(\frac{d^2P}{dx^2}-\frac{dQ}{dx}+R\right)\lambda=0\ldots(81),$$

a relation which we may denote by

$$F(\lambda)=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(82).$$

60. Let $m-n=1$; (80) becomes

$$P\lambda\frac{d^2\rho}{dx^2}+\left\{2\frac{dP}{dx}\lambda+2P\frac{d\lambda}{dx}-Q\lambda\right\}\frac{d\rho}{dx}+F(\lambda)\rho=0\ldots(83).$$

61. When (80) is soluble $\lambda$ is known, and, consequently, $u$; for $\rho$ and $n$ are arbitrary; and

$$t=\frac{P\lambda u}{\int R\lambda dx}.$$

And when $P$ is a constant (83) becomes

$$P\frac{d^2(\lambda\rho)}{dx^2}-Q\frac{d(\lambda\rho)}{dx}+\left(R-\frac{dQ}{dx}\right)\lambda\rho=0\ldots\ldots(84),$$

an equation resembling one to which the subject of cognate forms has conducted us; viz., equation (22) of Art. 12. If $\lambda\rho$ be determined from (84) $\rho$ is arbitrary, as before.

62. Let

$$R\left(\frac{dx}{dt}\right)^2=P\ldots\ldots\ldots\ldots\ldots\ldots\ldots(85),$$

or $$\frac{dx}{dt}=\sqrt{\left(\frac{P}{R}\right)}=\frac{1}{\sqrt{r}},\quad t=\int\sqrt{r}.dx.$$

Then if the last equation enables us to express $x$ as a function of $t$, so that we may write

$$t=\phi x,\quad x=\phi^{-1}t=\chi t,$$

$\phi$ and $\chi$ being recognized functions, the solution of (1) may be made to depend upon that of an equation of the form

$$\frac{d^2y}{dx^2}+2q\frac{dy}{dx}+y=0\ldots\ldots\ldots\ldots\ldots(86).$$

63. By a change of the independent variable (16) becomes

$$\frac{d^2y}{dt^2} + \left(2q\frac{dx}{dt} - \frac{dt}{dx}\frac{d^2x}{dt^2}\right)\frac{dy}{dt} + r\left(\frac{dx}{dt}\right)^2 y = 0 \ldots (87),$$

or, as we may write it,

$$\frac{d^2y}{dt^2} + 2\pi\frac{dy}{dt} + \rho y = 0 \ldots\ldots\ldots\ldots\ldots\ldots (88).$$

Assume $$2\pi + f\sqrt{(\rho)} + h = 0 \ldots\ldots\ldots\ldots\ldots\ldots (89).$$

which is equivalent to

$$2q\frac{dx}{dt} - \frac{dt}{dx}\frac{d^2x}{dt^2} + f\sqrt{(r)}\frac{dx}{dt} + h = 0,$$

or $$\frac{dt}{dx}\frac{d^2x}{dt^2} = \{2q + f\sqrt{(r)}\}\frac{dx}{dt} + h \ldots\ldots\ldots\ldots (90),$$

and we have $\log\frac{dx}{dt} = \int\{2q + f\sqrt{(r)}\}\,dx + \int hdt,$

which we may write in the form

$$\log\frac{dx}{dt} = \phi x + \chi t,$$

$h$ being regarded as an arbitrary function of $t$. Hence

$$\frac{dx}{dt} = e\phi^{x+\chi^t},$$

and $$\int\frac{dx}{e\phi^{x}} = \int e^{\chi^t}dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (91).$$

64. When $f$ and $h$ can be suitably assigned and $x$ can be expressed as a recognized, though transcendental, function of $t$, and *vice versâ*, (16) is transformable into

$$\frac{d^2y}{dt^2} + 2\pi\frac{dy}{dt} + \left(\frac{2\pi + h}{f}\right)^2 y = 0 \ldots\ldots\ldots\ldots (92),$$

which embraces a variety of cases.

65. To solve (15) by series: assume

$$y = \alpha_0 + \alpha_2 x^2 + \alpha_4 x^4 + \alpha_6 x^6 + \ldots = \Sigma\alpha_{2r}x^{2r}.$$

Then $$x\frac{dy}{dx} = \Sigma 2r\alpha_{2r}x^{2r},$$

$$x^2\frac{d^2y}{dx^2} = \Sigma 2r(2r-1)\,\alpha_{2r}x^{2r},$$

and (15) may be put under the form

$$\Sigma[(2r+2)(2r+1)\alpha_{2r+2}-\{2r(2r-1)+2r+b\}\alpha_{2r}]x^{2r}=0 \ldots (93).$$

Consequently, since the sinister must vanish independently of $x$,

$$\frac{\alpha_{2r+2}}{\alpha_{2r}}=\frac{(2r)^2+b}{(2r+2)(2r+1)} \ldots\ldots\ldots\ldots\ldots (94).$$

Again: assume

$$y=\beta_1 x+\beta_3 x^3+\beta_5 x^5+\ldots=\Sigma\beta_{2r+1}x^{2r+1}.$$

Then
$$x\frac{dy}{dx}=\Sigma(2r+1)\beta_{2r+1}x^{2r+1},$$

$$x^2\frac{d^2y}{dx^2}=\Sigma(2r+1)2r\beta_{2r+1}x^{2r+1},$$

and (15) may be put under the form

$$\Sigma[(2r+3)(2r+2)\beta_{2r+3}-\{(2r+1)^2+b\}\beta_{2r+1}]x^{2r+1}=0 \ldots (95),$$

and we have

$$\frac{\beta_{2r+3}}{\beta_{2r+1}}=\frac{(2r+1)^2+b}{(2r+3)(2r+2)} \ldots\ldots\ldots\ldots\ldots (96).$$

4, Pump Court, Temple, London,
*May* 21, 1862.

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# ON THE ANALOGY BETWEEN LINES OF CURVATURE AND PLANE CONICS.

By *J. G. Laing*, *B.A.*, Scholar of St. John's College, Cambridge.

Many properties of lines of curvature have been investigated which are analogous to properties of a plane conic; *some* of the following I think have not been noticed.

(I.) In an ellipse we have

$$p.r'=a.b \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where $p$ is the perpendicular on the tangent and $r'$ the semi-diameter parallel to the tangent.

In a line of curvature on an ellipsoid

$$p.D=\mu,\text{ a constant } \ldots\ldots\ldots\ldots\ldots (2),$$

where $p$ is the perpendicular on the tangent plane and $D$ the semi-diameter parallel to the tangent to the line of curvature.

(II.) In an ellipse

$$r'^2 = a^2 + b^2 - r^2,$$

therefore, by (1),

$$r^2 + \frac{a^2b^2}{p^2} = a^2 + b^2 \text{ ...................(3)},$$

the equation to the ellipse in terms of $r$ and $p$.

In a line of curvature

$$D^2 = a^2 + b^2 + c^2 - \lambda^2 - r^2,$$

where $\lambda$ is the constant semi-diameter parallel to the tangent to the perpendicular line of curvature; therefore, by (2),

$$r^2 + \frac{\mu^2}{p^2} = a^2 + b^2 + c^2 - \lambda^2 \text{ ...............(4)},$$

the equation to the line of curvature in terms of $r$ and $p$.

(III.) If $\rho$ be the radius of curvature at any point of an ellipse,

$$\rho = r\frac{dr}{dp} = \frac{a^2b^2}{p^3}, \text{ by (3)},$$

$$\rho = \frac{r'^2}{p}, \text{ by (1) ..........................(5)}.$$

If $\rho_1$ be the radius of curvature of a normal section of the ellipsoid, through the tangent to the line of curvature, then since two consecutive normals intersect, we have

$$\rho_1 = r\frac{dr}{dp} = \frac{\mu^2}{p^3}, \text{ by (4)};$$

therefore

$$\rho_1 = \frac{D^2}{p}, \text{ by (2) ........................(6)}.$$

(IV.) In an ellipse if $PG$ be the normal at $P$, $G$ being the point in which it meets the major axis

$$p \,.\, PG = b^2 \text{ .......................(7)}.$$

In an ellipsoid if $G$ be the point in which the normal meets the plane of $(xy)$ the coordinates of $G$ are $x - \frac{xc^2}{a^2}$, $y - \frac{yc^2}{b^2}$, 0; therefore

$$PG^2 = c^4 \left\{ \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} \right\},$$

$$= \frac{c^4}{p^2},$$

therefore $$p . PG = c^2 \text{ ..................................} (8).$$

(V.) In an ellipse

$$\rho = \frac{r'^2}{p} = \frac{a^2b^2}{p^3}, \text{ by (1)},$$

therefore $$\rho \propto PG^3, \text{ by (7) ....................} (9).$$

In a line of curvature

$$\rho_1 = \frac{D^2}{p} = \frac{\mu^2}{p^3} = \frac{\mu^2}{c^6} PG^3, \text{ by (8)};$$

therefore $$\rho_1 \propto PG^3 \text{ ......................} (10).$$

(VI.) In an ellipse, if the normal meet the minor axis in $G'$,

$$PG . PG' = r'^2 \text{ ....................} (11).$$

In a line of curvature if the normal meet the plane of $(yz)$ in $G_1$, that of $(zx)$ in $G_2$, and that of $(xy)$ in $G_3$, then, by (8),

$$PG_1 = \frac{a^2}{p}, \quad PG_2 = \frac{b^2}{p}, \quad PG_3 = \frac{c^2}{p};$$

therefore $$PG_1 . PG_2 = \frac{a^2b^2}{p^2} = \frac{a^2b^2}{\mu^2} D^2,$$

$$= \frac{\lambda^3}{a^2} . D^2 \text{ ......................} (12),$$

and $$PG_1 . PG_2 . PG_3 = \frac{a^2b^2c^2}{p^3} = \frac{a^2b^2c^2}{\mu^3} D^3,$$

$$= \frac{\lambda^3}{abc} D^3 = \lambda^2 \frac{D^2}{p},$$

$$= \lambda^2 \rho_1.$$

Through $P$ draw a tangent to the *other* line of curvature, and let it meet the principal planes in the points $T_1$, $T_2$, $T_3$; then, since this tangent is a normal to the ellipsoid

$$\frac{x^2}{a^2-\lambda^2}+\frac{y^2}{b^2-\lambda^2}+\frac{z^2}{c^2-\lambda^2}=1 \text{ ...........} (13),$$

and $P$ is a point on a line of curvature of this ellipsoid; therefore, by (12),

$$PT_2.PT_3=\frac{\lambda^2}{\lambda^2-a^2}D'^2 \text{ ...............} (14),$$

where $D'$ is the diameter of the ellipsoid (13) parallel to the tangent to the first line of curvature; also

$$PT_1.PT_2.PT_3=-\lambda^2\rho_2 \text{ ...............} (15),$$

$\rho_2$ being the radius of curvature of the normal section of the ellipsoid (13) by a plane through the tangent to the first line of curvature at $P$.

(*To be continued.*)

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# GEOMETRICAL CONICS.

By *C. Taylor, B.A.*, Scholar of St. John's College, Cambridge.

SINCE projection has not hitherto been admitted amongst the more elementary geometrical methods, it seems advisable to point out, that projective properties of the ellipse can be deduced very readily from properties of the auxiliary circles, if a few fundamental propositions be proved *once for all* and assumed in subsequent investigations. In the statements of Prop. I., II., III., IV., $SP$, $PS'$ denote lines drawn in directions from $S$ to $P$ and from $P$ to $S'$ respectively, so that the angles required to be equal are $gkP$, and the supplement of $gk'P$.

It may be remarked that the two corresponding results of Prop. IX. have been obtained by precisely similar methods, and that the proof of Prop. XII. applies also to the hyperbola.

Let $S$, $S'$ be the foci of an ellipse whose axes are $ACA'$, $BCB'$, and let the minor axis meet the circle described about the triangle $SPS'$ in $g$, where $P$ is any point on the curve.

Then, since $g$ is the middle point of the arc $SS'$, $Pg$ bisects the angle $SPS'$, and is therefore the normal at $P$. Also, if $t$ be the other extremity of the diameter through $g$ to the circle, then $tP$, being perpendicular to $Pg$, is the tangent at $P$. Let the tangent and normal meet the major axis in $T$, $G$.

PROP. I. *If from* $g$ (fig. 37) *straight lines* $gk$, $gk'$ *be drawn equally inclined to* $SP$, $PS'$, *then will*

$$kP + Pk' = AA'.$$

For, since $Sg = gS'$, and the angles $gSk$, $gkS$ equal the angles $gS'k'$, $gk'S'$ respectively, therefore $Sk = S'k'$; hence

$$kP + Pk' = SP + PS' = AA'. \quad \text{Q.E.D.}$$

COR. If $gk$ be perpendicular to $SP$, $Pk = Pk' = AC$.

N.B. In after propositions $gk$ will be considered perpendicular to $SP$.

PROP. II. *If* $O$ (fig. 38) *be the centre of the circle inscribed in the triangle* $SPS'$ *then*

$$PO : PG = CA : A'S.$$

Draw $OM$, $PN$ perpendicular to $SS'$, then

$$OM(SP + PS' + SS') = \text{twice the area of } SPS' = PN.SS',$$

but $OM : PN = OG : PG$ by similiar triangles, therefore

$$OG : PG = CS : CS + CA,$$

and $$PO : PG = CA : CS + CA. \quad \text{Q.E.D.}$$

PROP. III. *Straight lines* $Oy$, $Oy'$ (fig. 38) *are drawn from* $O$ *equally inclined to* $SP$, $PS'$, *to prove that*

$$yP + Py' = 2SA.$$

If $x$, $x'$ be the points of contact of $PS$, $PS'$ with the circle

$$Px = \tfrac{1}{2}(SP + PS' - SS') = CA - CS;$$

therefore $$Px = SA = Px'.$$

Also, in the right-angled triangles $xyO$, $x'y'O$, we have

$$\angle Oyx = \angle Oy'x',$$

and $$Ox = Ox';$$

therefore $$xy = x'y';$$

hence $$yP + Py' = xP + Px' = 2SA. \quad \text{Q.E.D.}$$

**PROP. IV.** *Straight lines $Gz$, $Gz'$ (fig. 38) are drawn from $G$ equally inclined to $SP$, $PS'$, show that*

$$zP + Pz' = \text{latus rectum}.$$

For convenience suppose $zG$, $z'G$ parallel to $yO$, $y'O$; then

$$zG + Gz' : yP + Py' = PG : PO,$$

by similar triangles, or

$$zG + Gz' : 2SA = A'S : CA,$$

(Prop. II., III.); therefore

$$zG + Gz' : 2CB = CB : CA,$$

which proves the proposition.

COR. 1. If $GK$ be perpendicular to $SP$, (fig. 37)

$$PK = \tfrac{1}{2} \text{ (latus rectum)}.$$

COR. 2. It is evident that

$$PG : Pg = PK : Pk = CB^2 : CA^2.$$

EXAMPLE. A circle touches an ellipse in two points, one of them being $P$, to show that its chord in direction $PS$ has one of two constant values.

If the points be on opposite sides of $BB'$ the chord is equal to $2Pk$, or the major axis, $g$ being the centre of the circle. Similarly, if they be on opposite sides of $AA'$, the chord is equal to $2PK$, that is, to the latus rectum.

Draw $Pn$ (fig. 40) perpendicular on the minor axis: then $CN$, $Cn$ may be conveniently distinguished as the *major* and *minor subnormals.*

Similarly it is sometimes convenient to call the circles on $AA'$, $BB'$ the *major* and *minor auxiliary circles.*

**PROP. V.** *The major and minor sub-normals at any point of an ellipse are to the distances of the point from the minor and major axes in the duplicate ratios of the semi-axes.*

For, by similar triangles,

$$NG : Pn = PG : Pg = CB^2 : CA^2,$$

(Prop. IV., Cor. 2), and, similarly,

$$ng : PN = Pg : PG = CA^2 : CB^2.$$

COR. Hence

$$CG : Pn = CA^2 - CB^2 : CA^2,$$

or $$CG : CN = CS^2 : CA^2;$$

so $$Cg : Cn = CS^2 : CB^2.$$

PROP. VI. *If kk′* (fig. 39) *meet Pg in F, the angles at F will be right angles.*

*Also kk′ passes through C.*

For $PF$ bisects the vertical angle of the isosceles triangle $kPk'$, which proves the first part.

Again, if $Sm$ be parallel to $S'P$, then

$$\angle Smk = \angle Skm,$$

for $\angle Smk' = \angle mk'S'$, and $\angle Pkk' = \angle Pk'k$.

Hence also $Sm = Sk = S'k'$, (Prop. I.), and, if $kk'$ meet $SS'$ in $C'$, the triangles $SC'm$, $S'Ck$ will be equal in all respects. Therefore $C'$ bisects $SS'$ and coincides with $C$, which proves the second part.

PROP. VII. *Every tangent to an ellipse meets its focal perpendiculars on a fixed circle.*

The lines $CY$, $CY'$ (fig. 39) drawn parallel to the focal distances, make equal angles with $Tt$, and are therefore equal. Also $SY$, $S'Y'$ are perpendicular to $PT$, and $CY = CY' = CA$. For, let $CY'$ cut $S'P$ in $O$. Then since a line through $C$ parallel to $S'P$ and meeting $SP$ is equal to $OP$ or $OS'$ and also $OP = OY'$ (Euc. I. 6), therefore $OP = OY' = OS'$, and $\angle S'Y'P$ is a right angle. Similarly $\angle SYP$ is a right angle.

Again $CO = \frac{1}{2}SP$ and $OY' = \frac{1}{2}S'P$, therefore

$$CY' = \tfrac{1}{2}(SP + S'P) = CA. \quad \text{Q.E.D.}$$

PROP. VIII. *To show that* (fig. 39)

$$PF.Pg = CA^2,$$

*and* $$PF.PG = CB^2.$$

Since $kF$ is perpendicular to $Pg$ and $gk$ to $Pk$ (Prop. VII.), we have

$$PF.Pg = Pk^2 = CA^2 \text{ (Prop. I., Cor.)},$$

therefore $$PF.PG = CB^2 \text{ (Prop. IV., Cor.)}.$$

**Prop. IX.** *To prove that* (fig. 40)

$$CN.CT = CA^2,$$

*and*

$$Cn.Ct = CB^2.$$

Let $PN$, $Pn$ meet $CF$ produced in $N'$, $n'$.

Then, since a circle can be described about $n'nFg$, the angles $n'ng$, $n'FG$ being right angles, therefore

$$Pg.PF = Pn.Pn' = CN.CT,$$

or

$$CN.CT = CA^2.$$

And since a circle can be described about $N'NGF$, therefore

$$PG.PF = PN.PN' = Cn.Ct,$$

or

$$Cn.Ct = CB^2.$$

**Prop. X.** *Straight lines drawn perpendicular to either axis of an ellipse are cut in a constant ratio by the circle on that axis and the curve.*

If $NP$ (fig. 41) meet the circle upon $AA'$ in $P'$, then

$$CN.CT = CA^2 = CP'^2 \text{ (Prop. IX.)},$$

therefore $TP'C$ is a right angle and $TP'$ *touches the circle.* Draw $Pn$ perpendicular to $BB'$ to meet $CP'$ in $p$, then by similar triangles,

$$PN : P'N = Cp : CP'.$$

Also

$$PN : P'N = Ct : CU,$$

if $TP'$ meet the minor axis in $U$. Therefore

$$Cp : CP' = Ct : CU,$$

and $tp$ *is parallel to* $UP'$ *or perpendicular to* $CP'$. We have then

$$Cn.Ct = Cp^2,$$

but

$$Cn.Ct = CB^2, \text{ (Prop. IX.)},$$

hence

$$Cp = CB.$$

It follows that

$$PN : P'N = CB : CA,$$

and

$$pn : Pn = CB : CA.$$

Also the circle on $BB'$ passes through $p$ and *is touched by* $tp$.

**Prop. XI.** *If* $CD$ (fig. 40) *be the semi-diameter conjugate to* $CP$, *then*

$$Pg : CD = CA : CB,$$

*and*

$$PG : CD = CB : CA.$$

For it may be shewn, as in Drew's *Conic Sections*, that

$$PF.CD = CA.CB,$$

but $$Pg.PF = CA^2, \text{ (Prop. VIII.)}$$

and $$PG.PF = CB^2,$$

therefore $$Pg : CD = CA : CB,$$

and $$PG : CD = CB : CA.$$

Cor. Hence $$PG.Pg = CD^2.$$

Prop. XII. *The rectangle contained by the focal distances of any point on the ellipse is equal to the square on the conjugate semi-diameter.*

For, by similar triangles, $SPg$, $S'PG$, (fig. 37) we have

$$SP : Pg = GP : PS',$$

therefore $SP.PS' = gP.PG = CD^2$. (Prop. XI., Cor.)

Also by similar triangles $tPg$, $TPG$,

$$tP.PT = gP.PG = CD^2.$$

Prop. XIII. *If* $SY$, $S'Y'$ (fig. 39) *be perpendicular to* $TP$, *then*

$$SY.S'Y' = CB^2.$$

For, by similar triangles, $PkF$, $PSY$, we have

$$SY : SP = PF : Pk = PF : CA,$$

and $$PF : CA = CB : CD, \text{ as above,}$$

therefore $$SY : SP = CB : CD.$$

So $$S'Y' : S'P = CB : CD.$$

Therefore $$SY.S'Y' : SP.S'P = CB^2 : CD^2,$$

but $$SP.S'P = CD^2,$$

therefore $$SY.S'Y' = CB^2. \quad \text{Q.E.D.}$$

Prop. XIV. *To find an expression for the central perpendicular on the tangent at any point to an ellipse.*

Let $CX'$ (fig. 39) be perpendicular to $PT$, and $S'Z'$ perpendicular to $CX'$. Then

$$CX'^2 = CY'^2 - XY'^2 = CA^2 - S'Z'^2. \text{ (Prop. VII.)}$$

Similarly, if $CX$ be the perpendicular on the tangent at right angles to $PT$, we see that

$$CX^2 = CA^2 - CZ'^2,$$

for $CZ' = SZ$, where $CZ$ is perpendicular to $SY$.

Cor. Hence we obtain proofs of the theorems, that *the intersection of tangents at right angles lies on a circle*, and that *the sum of the squares of the reciprocals of two diameters at right angles is constant.*

(*To be continued.*)

## QUERIES.

1. Can perfect numbers be expressed in any other form than that given in Wood's *Algebra*, p. 242, viz., $2^n(2^{n+1}-1)$, when $2^{n+1}-1$ is prime?

2. Wanted a formula to express all proper primes,* or a test to ascertain *à priori* whether a proposed prime is proper or not.

3. Wanted a construction for the following problem: To describe an isosceles triangle, having each angle at the base three times the vertical angle.—C.

---

* $P$ is a proper prime when $\frac{1}{P}$ enunciated into a circulating decimal produces a period of $P-1$ places.

END OF VOL. I.

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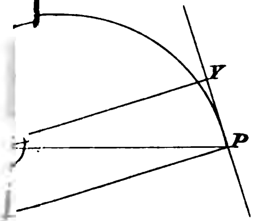

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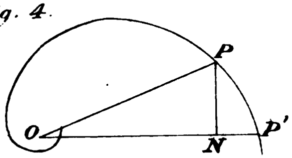

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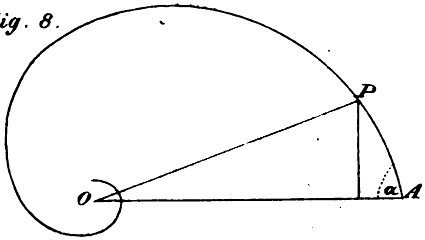

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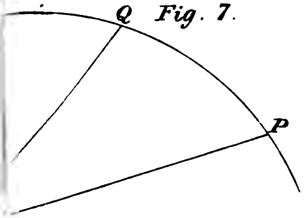

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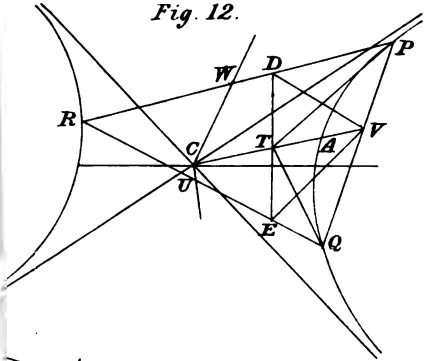

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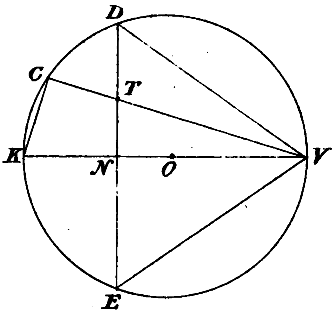

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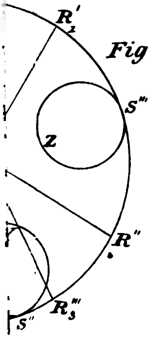

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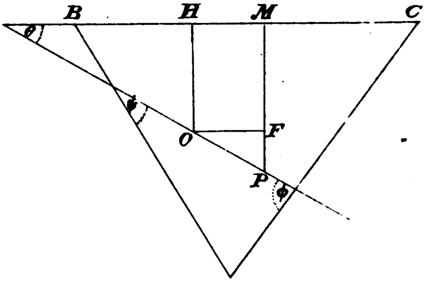

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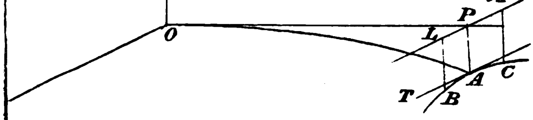

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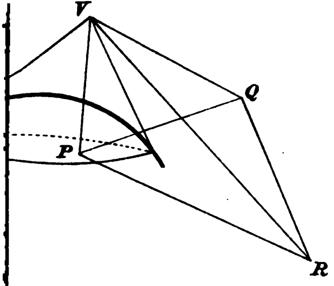
Fig. 20



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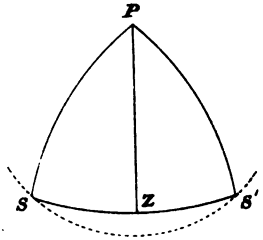

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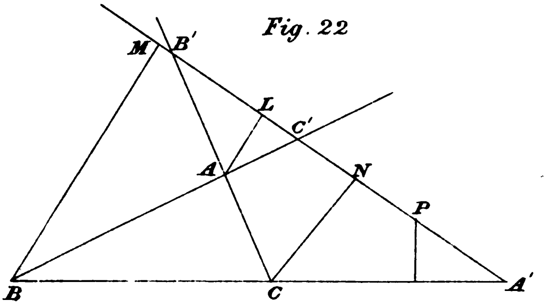

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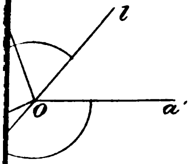

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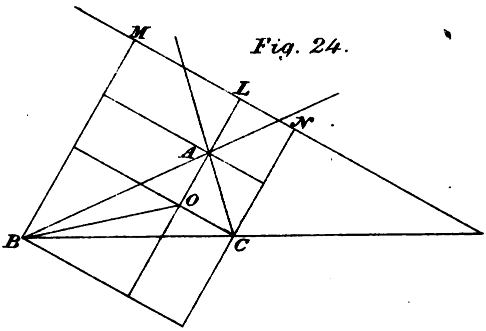

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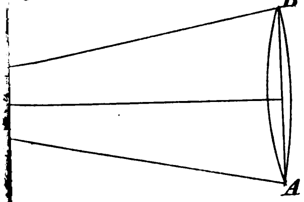

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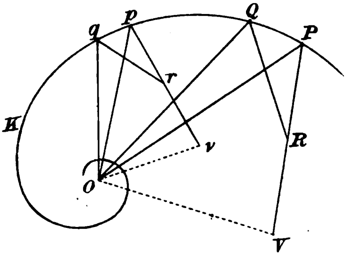

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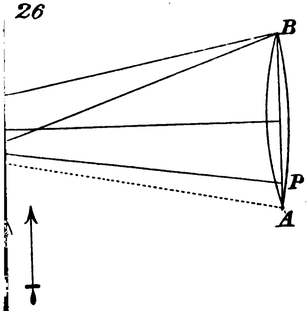

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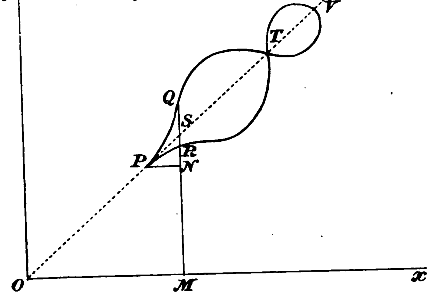

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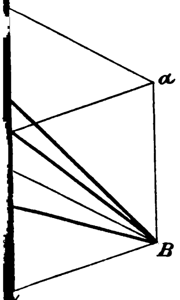

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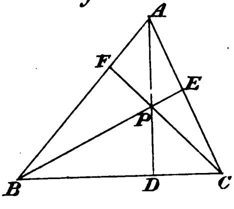

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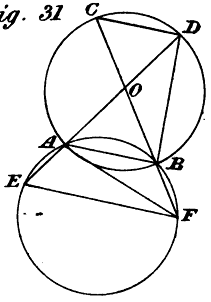

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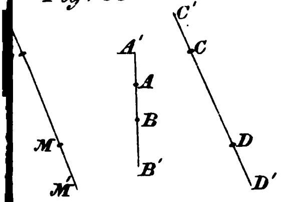

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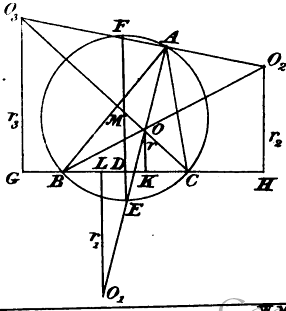

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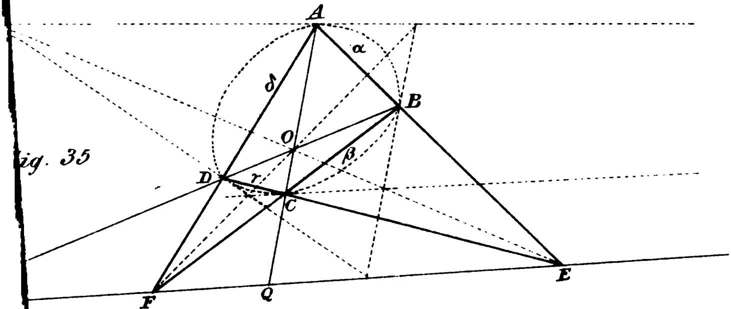

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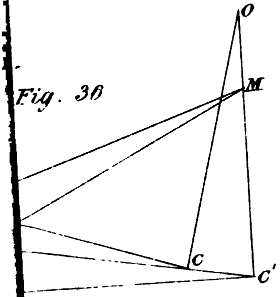

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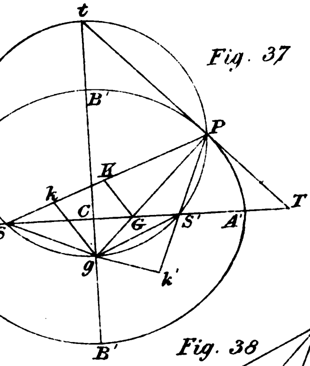

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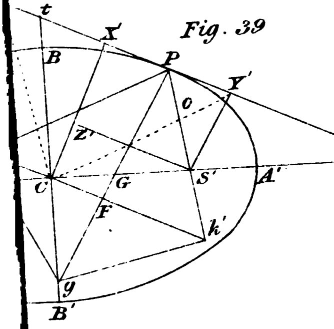

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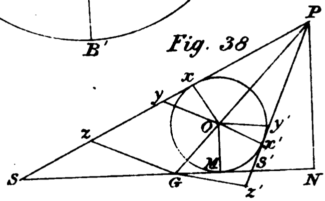

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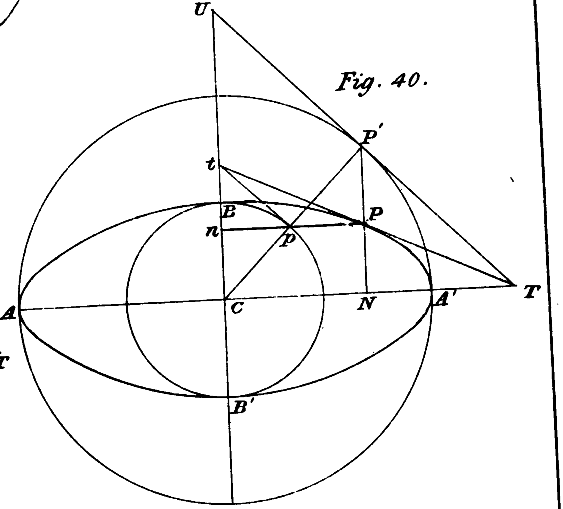

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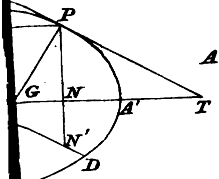

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